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Calculation of dynamo coefficients in cosmic-ray driven dynamo experiment

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Abstract

We present and discuss several new approaches of derivation of the dynamo transport coefficients for the cosmic-ray galactic dynamo experiment by Hanasz et al. (2004), in which we obtained amplification of the large-scale magnetic field on a timescale of 250 Myr. In this paper the standard description of hydromagnetic dynamo by Moffat (1978) is applied, modified and tested in case of the cosmic-ray driven dynamo model. For the first time $\hat{\alpha}$ and $\hat{\beta}$ tensors are derived using the multidimensional linear regression and the Levenberg-Marquardt minimization methods. We find that the dynamo coefficients strongly oscillate in the range of ± 10 km/s in the considered dynamo experiment.

1 Introduction

The classical turbulent dynamo theory (see Ferrière this volume) explains quite well the observed magnetic field in spiral galaxies (e.g. Otmianowska-Mazur et al., 2002) but there are still observational and theoretical constraints which are difficult to overcome (summarized in Widrow, 2002; see also Elstner this volume). The idea of the cosmic-ray driven fast dynamo proposed by Parker (1992) represents an attractive alternative for classical dynamos in galactic disks (see Hanasz et al., 2004 and Hanasz et al. this volume). The dynamo model described in the mentioned papers relies on the buoyancy of cosmic rays in galactic disks, on the differential rotation, rotational pseudoforces and resistivity of the ISM. In this model cosmic rays are injected into the ISM in randomly distributed supernova remnants. Amplification of the mean magnetic field on a timescale of 250 Myr has been observed during the period of about 2 Gyr. The aim of this contribution is to present computations of the dynamo transport coefficients for the cosmic-ray driven dynamo simulations by Hanasz et al. (2004).

2 Equation of dynamo

The dynamo equation results from the induction equation as a result of decomposition of the velocity and magnetic fields into mean and fluctuating parts, where an averaging of statistical ensemble, spacial domain or time is performed. The standard form of the dynamo equation is (Moffat, 1978),

$$\frac{\partial^{=}\overline{\mathbf{B}}}{\partial t^{=}}\nabla \times \left(\overline{\mathbf{V}} \times \overline{\mathbf{B}} + \mathcal{E} + \eta \nabla \times \overline{\mathbf{B}}\right), \tag{1}$$

where **B** and **V** means the large-scale velocity and magnetic fields. The electromotive force $\mathcal{E}$ is represented in the form proposed by Rädler (1980)

$$\mathcal{E} = \overline{\mathbf{v} \times \mathbf{b}} = -\alpha \cdot \overline{\mathbf{B}} - \gamma \times \overline{\mathbf{B}} - \beta \cdot \left(\nabla \times \overline{\mathbf{B}}\right) - \delta \times \left(\nabla \times \overline{\mathbf{B}}\right) , \tag{2}$$

where **v** and **b** means the small-scale velocity and magnetic fields and α-term corresponds to generalization of the α-effect, γ-term describes a transport of mean magnetic flux, and β-term and δ-term are symmetric and antisymmetric parts of the diffusion tensor, respectively.

Each component of $\mathcal{E}$ is expanded into Taylor series and approximated as

$$\mathcal{E}_{\mathrm{i}} = a_{\mathrm{ij}} \overline{B}_{\mathrm{j}} + b_{\mathrm{ijk}} \frac{\partial \overline{B}_{\mathrm{j}}}{\partial x_{\mathrm{k}}} . \tag{3}$$

Relation between representation of the coefficients in Eq. (2) and the expansion coefficients in Eq. (3) is straightforward (Rädler, 1980):

$$\begin{array}{ll} \alpha_{\mathrm{ij}} = -\frac{1}{2}\left(a_{\mathrm{ij}} + a_{\mathrm{ji}}\right), & \gamma_{\mathrm{i}} = +\frac{1}{2}\epsilon_{\mathrm{ijk}} a_{jk}, \\ \beta_{\mathrm{ij}} = +\frac{1}{4}\left(\epsilon_{\mathrm{ikl}} b_{\mathrm{jkl}} + \epsilon_{\mathrm{jkl}} b_{\mathrm{ikl}}\right), & \delta_{\mathrm{i}} = -\frac{1}{4}\left(b_{\mathrm{kik}} - b_{\mathrm{kki}}\right) . \end{array} \tag{4}$$

3 Methods of computations of the dynamo coefficients

The computation of the α, β, γ and δ tensors is performed in three steps. The first step is computation of electromotive forces on the base of simulation results, second step is fitting of the $\mathcal{E}_{\mathrm{i}}$ functions given by the relation (3) with the aid of one of the three methods described below. The last step is computation of the helicity and diffusivity tensor components through symmetric and antisymmetric transformations (4). In our analysis we use two types of averaging. The first one is averaging over XY-planes, so we have only Z-dependence, similarly to averaging used in Brandenburg-Sokoloff method (2002). The second one is based on averaging of magnetic field components in the cubes $\Delta x \times \Delta y \times \Delta z$ centered at given point (x, y, z).

3.1 Brandenburg-Sokoloff method (BS)

Brandenburg and Sokoloff (2002) applied the following method to determine the turbulent transport coefficients. They reduced the equations (3) to only four terms due to the plane averaging and due to vanishing of the vertical component of the mean magnetic field,

$$\begin{aligned} \mathcal{E}_x &= a_{xx}\overline{B}_x + a_{xy}\overline{B}_y + b_{xxz}\overline{B}_x' + b_{xyz}\overline{B}_y' \\ \mathcal{E}_y &= a_{yx}\overline{B}_x + a_{yy}\overline{B}_y + b_{yxz}\overline{B}_x' + b_{yyz}\overline{B}_y' \end{aligned} \tag{5}$$

If we define vectors $\mathsf{B} = \left[\overline{B}_x\, \overline{B}_y\, \overline{B}_x'\, \overline{B}_y'\right]$, $\mathsf{C}^{\mathrm{i}} = [a_{ix}\, a_{iy}\, b_{ixz}\, b_{iyz}]$, and matrices $\mathsf{M} = \langle \mathsf{B}^{\mathrm{T}}\mathsf{B}\rangle$, $\mathsf{E}^{\mathrm{i}} = \langle \mathcal{E}_{\mathrm{i}} \mathsf{B}^{\mathrm{T}}\rangle$, then each equation in (5) can be rewritten in the matrix form

$$\mathsf{E}^{\mathrm{i}} = \mathsf{M}\mathsf{C}^{\mathrm{i}}, \qquad \mathrm{i} = x,\, y\, . \tag{6}$$

The solution for unknown vector C^{i} is $\mathsf{C}^{\mathrm{i}} = \mathsf{M}^{-1}\mathsf{E}^{\mathrm{i}}$. We extend BS method by adding constant terms c_x and c_y additionally to Eqs. (5), so Eq. (6) is now $\mathsf{E}^{\mathrm{i}} = \mathsf{M}\mathsf{C}^{\mathrm{i}} + \mathsf{c}^{\mathrm{i}}$. We denote extended version by BS/const.

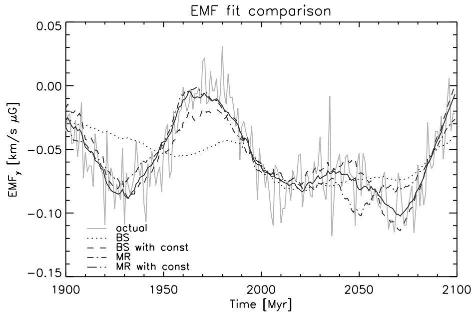

Figure 1: Comparison of the results obtained with the help of different methods of fitting for the chosen period of $\mathcal{E}_y$ time evolution at the height of $+400$ pc.

method	a_{yx}	a_{yy}	a_{yz}	b_{yxz}	b_{yyz}	b_{yzz}	c_y	χ^2
BS	-1.2±0.3	-0.11±0.06	-	31±5	1±3	-	-	0.163
BS/const	1.3±0.3	0.84±0.07	-	31±3	40±4	-	-1.11±0.07	0.073
MR	-0.4±0.2	0.21±0.05	7.4±0.6	-52±11	-21±3	50±16	-	0.077
MR/const	1.0±0.3	0.72±0.08	3.4±0.8	2±12	15±6	51±14	-0.70±0.10	0.059

Table 1: Coefficients with standard deviations derived in different fitting methods for plane averaging. Units of a_{ij}, b_{ijk} and c_i coefficients are [km/s], [10^{23}cm^2/s], and [km/s μG], respectively.

3.2 Multidimensional linear regression (MR)

We assume the linear relation of the quantity Y with m independent variables X_m:

$$\mathsf{Y} = \mathsf{a}_1\mathsf{X}_1 + ... + \mathsf{a}_\mathrm{m}\mathsf{X}_\mathrm{m} + \mathsf{a}_{\mathrm{m}+1}\,. \tag{7}$$

In the case of the second order approximation Y represents the vector of n different and independent samples of $\mathcal{E}_\mathrm{i}$. X_m are similarly, n-component vectors containing n samples of $\overline{\mathbf{B}}_\mathrm{j}$ and their spatial derivatives $\partial_{x_\mathrm{k}}\overline{\mathbf{B}}_\mathrm{j}$. Let us stress, that we include also a constant term $\mathsf{a}_{\mathrm{m}+1}$, which is independent on the mean magnetic field components and their spatial derivatives.

Eq.(7) rewritten in the matrix form $\mathsf{Y} = \mathsf{XA}$ with the vector of coefficients A, has solution of the form,

$$\mathsf{A} = \left(\mathsf{X}^\mathrm{T}\mathsf{X}\right)^{-1}\mathsf{X}^\mathrm{T}\mathsf{Y}. \tag{8}$$

3.3 Verification of the fitting quality

We calculate standard deviations of fitted coefficients using the following formula

$$\sigma^2_\mathsf{A} = \mathrm{diag}\left(\mathsf{X}^\mathrm{T}\mathsf{X}\right)^{-1}\chi^2/(k-n)\,, \tag{9}$$

where $\chi^2 \equiv \sum_n \left[\mathsf{y}(\mathsf{x}_{1n},\ldots,\mathsf{x}_{kn}) - \mathsf{y}_n\right]^2$ is a measure of fitting quality (Press et al., 1994).

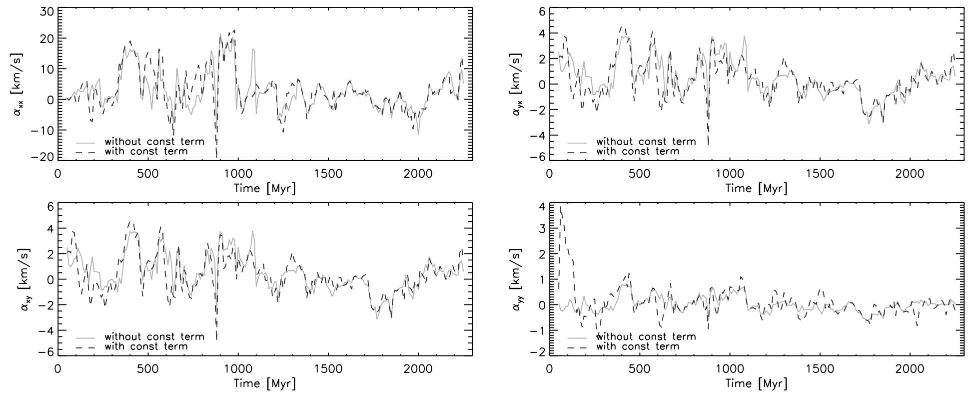

Figure 2: The time evolution of α-coefficients at z=200pc for the plane averaging method.

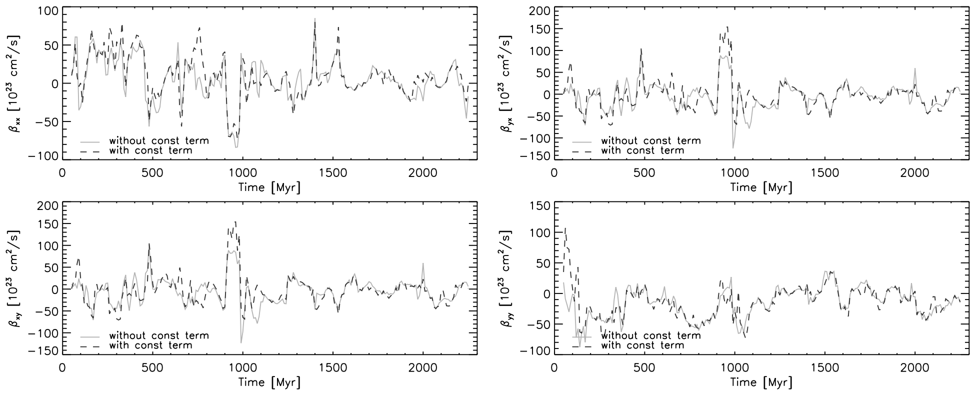

Figure 3: The time evolution of β-coefficients at z=200pc for the plane averaging method.

4 Results

4.1 Plane averaging

We perform calculations of the dynamo coefficients by applying the methods described in Section 3 with the aid of the constant-z averaging in the period $1900 \leq t \leq 2100$ Myr. First, we use the BS method (Eq. 6) to fit coefficients a_{ix}, a_{iy}, b_{ixz} and b_{iyz}. Then we apply BS/const method and fit the same coefficients along with the constant term. Subsequently we apply the MR and MR/const methods.

In Fig. 1 we present evolution of the $\mathcal{E}_y$ calculated with four methods at $z = 400$ pc. We notice that the best fitting curve is obtained with the aid of MR/const method (the double-dot-dashed line). The next two methods: BS/const (the dashed line) and MR (the dot-dashed line) fit quite well $\mathcal{E}_y$ and the BS method (the dotted line) leads to the largest deviations. The averaged values of coefficients contributing to $\mathcal{E}_y$ and the corresponding χ^2 are displayed in Table 1. We find that, in the case of coefficients a all the fitting methods give significantly different values, which are even of the opposite sign in cases of BS and MR methods.

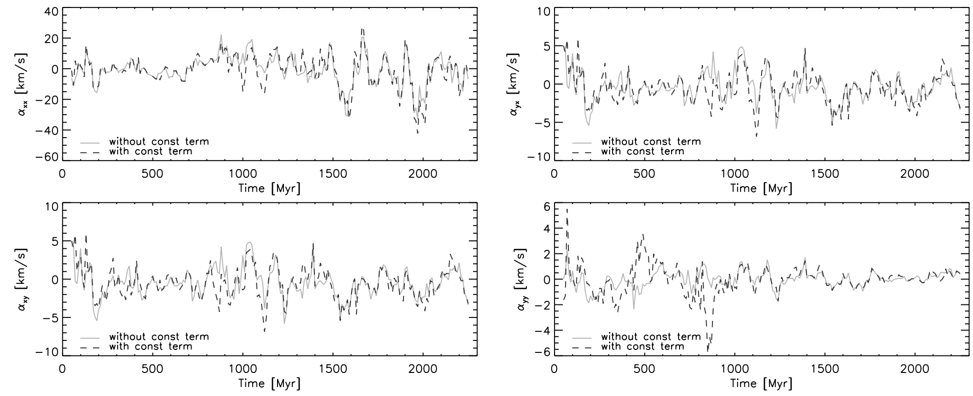

Figure 4: The time evolution of α-coefficients at z=200pc for the cube averaging method.

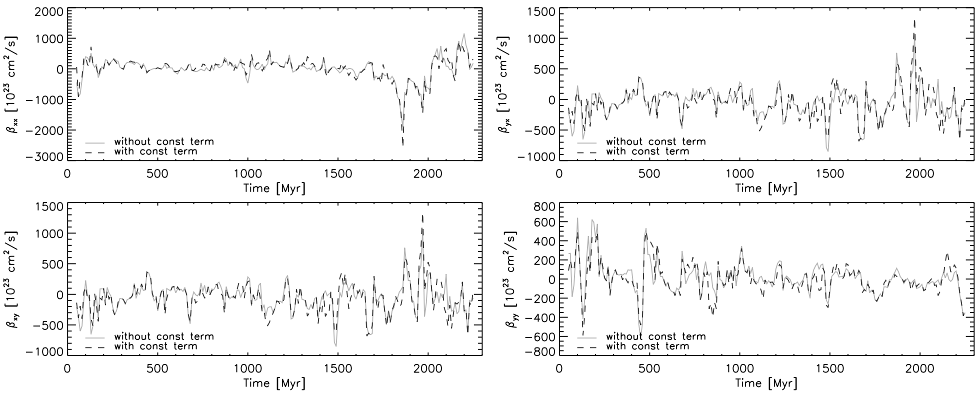

Figure 5: The time evolution of β-coefficients at z=200pc for the plane averaging method.

The presence of the constant term improves the fitting quality since χ^2 decreases. In the case of MR/const method χ^2 is about 3 times smaller when compared to results from the BS method. The values of diffusion coefficients are significantly different and their signs resulting from the different methods are different.

Figs. 2 and 3 show the time evolution of the selected coefficients for the MR method calculated at the height of 400 pc with and without the constant term. Generally both the curves are close to each other, although differences of the order of a few km/s are apparent. The amplitude of α_{xx} variations reaches the level of about ± 20 km/s, while of the remaining α coefficients are about ± 4 km/s. The coefficients β variate between ± 150 [10^{23} cm^2/s reaching occasionally negative values.

4.2 Cube averaging

The cube averaging better fulfills the assumption, imposed by the dynamo theory, that $\overline{B}$ should vary only weakly in a space and that the spatial derivatives of $\overline{B}$ of the second and higher orders should be negligible (Rädler 1980).

The MR method based on this kind of averaging makes it possible to determine all components of the α and β tensors, although the vanishing mean value of B_z implies that some components of the helicity and diffusivity tensors vanish.

In this case, α-coefficients vary in the similar range (Fig. 4) as in case of plane averaging, but their time-evolution is much different. In Fig. 5 we notice, however, that β-coefficients oscillate in 10 times wider range than in case of plane averaging.

5 Summary and Conclusions

We presented computations of the dynamo coefficients for the cosmic-ray driven dynamo, based on the MR method. Proposed method gives us statistical parameters describing quality of fitting. We performed also calculations of the dynamo coefficients using Brandenburg and Sokoloff (2002) local method for comparison. We applied several modifications to the BS method, which are aimed to overcome deficiencies of the original method. In order to evaluate correctness of each approach we compute the χ^2 residuals for the difference between the original and the reconstructed electromotive force. We have found that:

1. The original local BS method leads to a poor reconstruction of the EMF.
2. The BS method with the additional constant term provides a much better reconstruction of the original EMF, although we are unable to propose a physical interpretation of this term.
3. The MR method without the constant term leads to the comparable results as BS with a constant term.
4. The best reconstruction of EMF, although only slightly better than the previous one, results from the MR method with the constant term taken into account.
5. Each approach (even the last three ones, which are characterized by the same EMF reconstruction quality) lead to significantly different dynamo coefficients.

A further investigation of the cosmic-ray driven dynamo and its formal description in terms of the dynamo tensors $\hat{\alpha}$ and $\hat{\beta}$ is thus required.

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