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Superluminous accretion discs

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Summary. In this paper I compute upper limits for the total luminosities and collimation of radiation from thick, radiation supported accretion discs around black holes. I present numerical results obtained for the 'extreme' discs with $r_{\text{out}} = 10^3 GM_{\text{BH}}/c^2$, the angular momentum of the black hole being $J_{\text{BH}} = 0.998\, GM_{\text{BH}}/c$. The high luminosity ($L \sim 8.5\, L_{\text{Edd}}$) and substantial collimation of radiation found for these discs indicate that such discs can explain both the high luminosities of quasars and similar objects and may produce some of the observed beams and jets.

1 Introduction

The theory of thick accretion discs has recently been developed in both Newtonian (Paczyński & Wiita 1980 = PW) and general relativistic (Jaroszyński, Abramowicz & Paczyński 1980 = JAP) cases. In these papers, the whole uncertainty connected with our poor knowledge of such physical properties as the mechanisms of viscosity, energy transport, convection, etc. is summarized in just one free function: the surface distribution of the angular momentum. Assuming a specific form for this, one can compute not only the shape of the disc but also the properties of the radiation with no references to the disc interior at all. Neither PW nor JAP took into account the reflection effect, i.e. absorption and re-emission of radiation in the surface layer of the disc. Most of the luminosity in a thick disc is radiated from the surface of two funnels formed along the rotation axis, so reflections are very important and are included in my numerical calculations.

The computer code I developed and used here has application for calculations of the radiation field on the surface of the disc and at infinity. In this paper I describe briefly the theory (Section 2) and numerical methods (Appendix A) on which such calculations are based. In Section 3 results are presented of calculations for the special case of JAP's model, for the thickest discs.

The high luminosity ($L \sim 8.5\, L_{\text{Edd}}$) and strong collimation of radiation along the rotation axis, found for extremely thick discs with fixed $r_{\text{out}} = 10^3 GM_{\text{BH}}/c^2$, indicate that thick

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accretion discs around supermassive black holes can explain the high luminosities of quasars and similar objects and may be able to accelerate and collimate some of the observed beams and jets. This idea was suggested by Lynden-Bell (1978) and developed by Paczyński (PW and JAP) and his co-workers.

2 Theory

2.1 THE SHAPE OF THE DISC

Following PW and JAP, I shall assume that a non-self-gravitating disc orbiting a Kerr black hole is steady and axisymmetric, and is built from a perfect fluid matter in mechanical equilibrium. The disc is thick for $r < r_{\text{out}}$ and it is Keplerian beyond r_{out}. The inner edge is located between the marginally bound orbit and marginally stable orbit (Abramowicz, Jaroszyński & Sikora 1978; Kozlowski, Jaroszyński & Abramowicz 1978). From JAP (equation A.13) the shape of the disc surface is given by

$$\frac{d\theta}{dr} = -\frac{l^2\,\nabla_r\,(g_{tt}/\rho^2) + 2l\,\nabla_r\,(g_{t\phi}/\rho^2) + \nabla_r\,(g_{\phi\phi}/\rho^2)}{l^2\,\nabla_\theta\,(g_{tt}/\rho^2) + 2l\,\nabla_\theta\,(g_{t\phi}/\rho^2) + \nabla_\theta\,(g_{\phi\phi}/\rho^2)}, \tag{1}$$

with boundary conditions

$$l_{\text{in}} = l_{\text{Kepler}}\,(r_{\text{in}})$$

and

$$l_{\text{out}} = l_{\text{Kepler}}\,(r_{\text{out}}).$$

The distribution of angular momentum should satisfy two additional conditions $dl/dr \geqslant 0$ – according to stability criteria and $d\Omega/dr < 0$ – because the angular momentum has to be transported outward to make accretion possible.

If the mass of the black hole and r_{out} are fixed, then the thickest and most luminous discs have the extreme distribution of angular velocity and momentum (JAP; Abramowicz, Calvani & Nobili 1980)

$$\begin{aligned} \Omega &= \text{const} = \Omega_{\text{in}} \quad \text{for} \quad r_{\text{in}} \leqslant r \leqslant r_{c_1}, \\ l &= \text{const} = l_c \quad \text{for} \quad r_{c_1} \leqslant r \leqslant r_{c_2}, \\ \Omega &= \text{const} = \Omega_{\text{out}} \quad \text{for} \quad r_{c_2} \leqslant r < r_{\text{out}}, \end{aligned} \tag{2}$$

where l_c can be derived from JAP (equation A.19).

2.2 THE DISTRIBUTION OF THE RADIATION FIELD ON THE DISC SURFACE

Following PW and JAP, I adopt the assumption that the net flux of radiation on the surface of the disc is 'critical' in the sense that the radial pressure-gradient force balances the effective gravity, g_{ef}, the latter being a composition of gravitational and centrifugal acceleration. Of the reflection effect I shall assume that the absorbed radiation is re-emitted isotropically in the rest frame of the surface element of the disc. Then the balance of forces in the direction normal to the surface will be achieved if

$$F_e - F_{\text{abs}}^{(\text{N})} = g_{\text{ef}}/\kappa, \tag{3}$$

where κ is the opacity per unit mass, $F_{\text{abs}}^{(\text{N})}$ is the normal component of the absorbed flux and F_e is the sum of the flux generated in the disc and the flux re-emitted isotropically in the surface layer of the disc. The tangent components of the absorbed radiation are left unbalanced, but I assume that their influence on the shape of the disc is negligible*. Taking into account the following relations, known from geometrical optics (JAP; Misner, Thorne & Wheeler 1973; Cunningham & Bardeen 1973)

$$T^{(i)(k)} = \int_\Omega I_0 n^{(i)} n^{(k)} d\Omega,$$

$$I_0 = I_e g^4$$

and equation (A.1), the normal component of the absorbed flux can be expressed as

$$F_{abs}^{(N)} \equiv T^{(t)(N)} = \frac{1}{\pi}\int_\Omega g^4 F_e n^{(N)} d\Omega, \tag{4}$$

where Ω is the solid angle in which the observer sees the source of radiation and

$$g = \frac{\nu_{obs}}{\nu_{em}} = \frac{1}{1+z}.$$

Substituting equation (4) in equation (3) we obtain an integral equation for F_e. Dividing the disc into a finite number of r-rings with quasi-uniform surface properties we can rewrite our equation in the approximate version

$$(F_e)_k - \sum_l B_k^l (F_e)_l \cong (g_{ef})_l/\kappa \tag{5}$$

with

$$B_k^l = \frac{1}{\pi}\int_{\Omega(k,l)} g^4 n^{(N)} d\Omega, \tag{6}$$

where $\Omega(k,l)$ denotes the solid angle in which the observer located at the kth ring sees the lth ring. Elements B_k^l are computed numerically by the method presented in Appendix A, whereas values for $(g_{ef})_l$ result directly from JAP equation (A.12). As a solution of the system of algebraic linear equations (5) we obtain the distribution of the surface flux, F_e. Computing the following elements

$$(BF)_k^l = \frac{1}{\pi}\int_{\Omega(k,l)} g^4 n^{(\phi)} d\Omega$$

and

$$(BS)_k^l = \frac{1}{\pi}\int_{\Omega(k,l)} g^4 n^{(S)} d\Omega$$

we can find tangent components of the absorbed flux, $F_{abs}^{(\phi)}$ and $F_{abs}^{(S)}$ using the approximate formulae:

$$(F_{abs}^{(\phi)})_k \equiv (T_{abs}^{(t)(\phi)})_k \cong \sum_l (BF)_k^l (F_e)_l$$

and

$$(F_{abs}^{(S)})_k \equiv (T_{abs}^{(t)(S)})_k \cong \sum_l (BS)_k^l (F_e)_l.$$

*Consistency of this assumption with the model was partially tested by me and results are discussed in Section 3.

2.3 THE DISTRIBUTION OF THE EFFECTIVE LUMINOSITY AT INFINITY

The effective luminosity at infinity is defined as:

$$L_{\rm ef}(\theta,\phi)=4\pi\,\frac{\delta L}{\delta\Omega(\theta,\phi)}{}^{\star}, \tag{7}$$

where θ and ϕ are in our case Boyer–Lindquist coordinates and δL is the energy which reaches a distant (θ,ϕ)-observer in the infinitesimal solid angle, $\delta\Omega$. As we have an axisymmetric case, it is useful to divide the angle θ into finite number of segments. Then the average effective luminosity in the ith segment can be expressed as follows:

$$(L_{\rm ef})_i=\frac{4\pi}{\Delta\Omega_i}\sum_k C_i^k\,\mathscr{L}_k, \tag{8}$$

where $\Delta\Omega_i=2\pi(\cos\theta_i-\cos\theta_{i+1})$, $\mathscr{L}_k$ is the energy 'at infinity' emitted from the kth disc ring per unit time, t, and C_i^k is that part of this energy, which reaches infinity in the kth segment. From JAP equation (A.33a)

$$\mathscr{L}_k=2\pi\int_{r_k}^{r_{k+1}}UF_{\rm e}\,\rho\left[g_{rr}+g_{\theta\theta}\left(\frac{d\theta}{dr}\right)^2\right]^{1/2}dr, \tag{9}$$

and from equation (A.3) of this paper

$$C_i^k=\frac{\displaystyle\int_{\Omega(i,k)}g_\infty^2\,n^{(\rm N)}\,d\Omega}{\displaystyle\int_{2\pi}g_\infty^2\,n^{(\rm N)}\,d\Omega}, \tag{10}$$

where $\Omega(i,k)$ denotes the solid angle measured in the emitter rest frame located in the kth disc ring, into which the photons reaching infinity in the ith segment are emitted and $g_\infty=\nu_\infty/\nu_{\rm em}$. Derivation of the above formula as well as the numerical method for calculations of C_i^k are presented in the Appendix A.

2.4 PHOTON TRAJECTORIES

Computations of the elements B_k^l and C_i^k require numerical calculations including accurate tracing of thousands of photons. This can be done if it is assumed that the funnels are void or filled with optically thin matter. Then trajectories of photons will be null-geodesies, which are described in the case of the Kerr spacetime by the system of four first-order ordinary differential equations (Misner *et al.* 1973). For axisymmetric and steady-state discs the calculations simplify immensely, since it is unnecessary to consider the motion in coordinates ϕ and t. Thus, the problem will reduce to one equation:

$$\frac{d\cos\theta}{dr}=\pm\sqrt{\frac{G(\cos\theta)}{F(r)}}, \tag{11}$$

where

$$G(\cos\theta)=-a^2E^2\cos^4\theta-(\Phi^2+Q-a^2E^2)\cos^2\theta+Q,$$

$$F(r)=E^2r^4-(\Phi^2+Q-a^2E^2)r^2+2[(aE-\Phi)^2+Q]\,r-a^2Q,$$

★ This is the total luminosity of the source that will be deduced by the distant (θ,ϕ)-observer, if he assumes that the source is shining isotropically.

whereas E, Φ and Q are constants of motion, which can be expressed as a function of the components of photon four-momentum (Misner *et al.* 1973).

3 Results

I chose for my calculations extreme discs i.e. ones with '$\Omega - l - \Omega$' and '$l - \Omega$' surface radiation laws (see equation 2 and Fig. 1), orbiting a black hole spinning with angular momentum $J_{\rm BH} = 0.998\,GM_{\rm BH}^2/c$ (Thorne 1974). In this paper I present the results only for discs with $r_{\rm out} = 10^3 GM_{\rm BH}/c^2$, but in a subsequent paper I shall review the radiation features for extreme discs with different $r_{\rm out}$. As it was shown by JAP, for $r_{\rm out}$ fixed the luminosity of '$\Omega - l - \Omega$' discs increases very slowly with increasing accretion rate, $\dot{M}$ (curve 2 in JAP Fig. 5) which results from the fact that the shapes of this type of disc are very similar for different values of $r_{\rm in}$. Therefore I confined my calculations only to two marginal cases i.e. with $r_{\rm in} = r_{\rm ms}$ and $r_{\rm in} \approx r_{\rm mb}$. As it is seen from Table 1 obtusenesses of both such discs and luminosities generated by them differ very little. However, significant differences occur in the distribution of the surface flux near the inner edge (Fig. 2), but radiation emitted in this region contributes weakly to the luminosity at infinity (Fig. 5). In consequence, the luminosities observed by distant observer and collimations of the radiation are practically the same for two extreme positions of $r_{\rm in}$. Constructing the disc by JAP's method I had to assume that radiation filling the funnels does not influence the shape of the disc. Consistency of this assumption with the model can be verified only *a posteriori* from the knowledge of the surface distribution of radiation exerting pressure and shear on the walls of the funnels. It can be seen from Fig. 3, that the disc is most susceptible to destruction by external pressure at its highest parts, this resulting from the fact that the decrease of the external pressure with increasing distance from the black hole is smaller than the same decrease of the inner pressure just below the surface of the disc. Therefore central pressure (equation B2) must be sufficiently high to make the whole disc resistant to the external pressure. On the other hand, the central pressure cannot be too high without violating the assumption about negligible self-gravity effects (equation B5). Then, the central pressure must lie between certain limits (Fig. 4) in order to maintain consistency with the above assumptions. The role of the tangent components of the absorbed radiation is not clear and it may be different in different parts of the disc. Relatively big $F_{\rm abs}^{(S)}$ components in higher parts of the disc may pull out matter from the surface layer causing a disc wind. In these parts the role of components $F_{\rm abs}^{(\phi)}$ is negligible as $F^{(\phi)} \ll g_{\rm ef}/\kappa$. In the inner region the components of radiation are

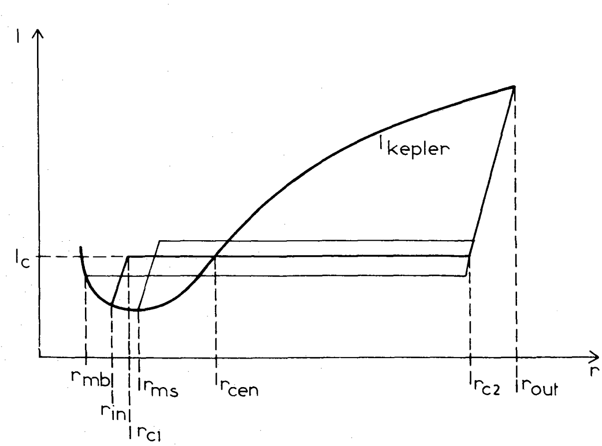

Figure 1. The distribution of angular momentum for a family of discs with fixed $r_{\rm out}$ (no scale).

Table 1. The numerical values describing the main features of a thick accretion disc and its radiation. The results presented here are obtained for discs with $r_{out} = 10^3 GM_{BH}/c^2$, the angular momentum of the black hole being $J_{BH} = 0.998\ GM_{BH}^2/c$.

Type of the distribution of the angular momentum	$r_{in}/(GM/c^2)$	Obtuseness of the disc's funnels ($2\times\theta_{min}$)	Efficiency $(1-U)$	Total luminosity generated by the disc ($\mathscr{L}/L_{Edd}$)	Accretion rate ($\dot{M}\times 1\,yr/M$)	Total luminosity reaching infinity (L/L_{Edd})	Collimation $(\theta_{col})^{\star}$
$l-\Omega$	1.091516 $r_{mb}=1.091443$	12°.8	0.001432	12.2	1.78×10^{-5}	8.5	25°
$\Omega-l-\Omega$	1.236971 $=r_{ms}$	13°.2	0.320994	11.8	7.67×10^{-8}	8.5	25°

$^{\star}\theta_{col}$ is defined as an angle corresponding to the solid angle $\Delta\Omega = 2\times 2\pi\cos\theta_{col}$ ($2\times$ – because two funnels) in which half of L is reaching infinity.

very big and exert strong torque on the disc, which can cause redistribution of the angular momentum and change the shape of the inner parts of the disc. Accurate treatment of this problem is very important in order to establish the position of the inner edge and therefore $\dot{M}$, but it will be possible to do it only in future models of thick discs based on a full theory of their interior. Final results, i.e. the distribution of the luminosity at infinity and the contribution of different parts of the disc to this distribution, are presented in Fig. 5.

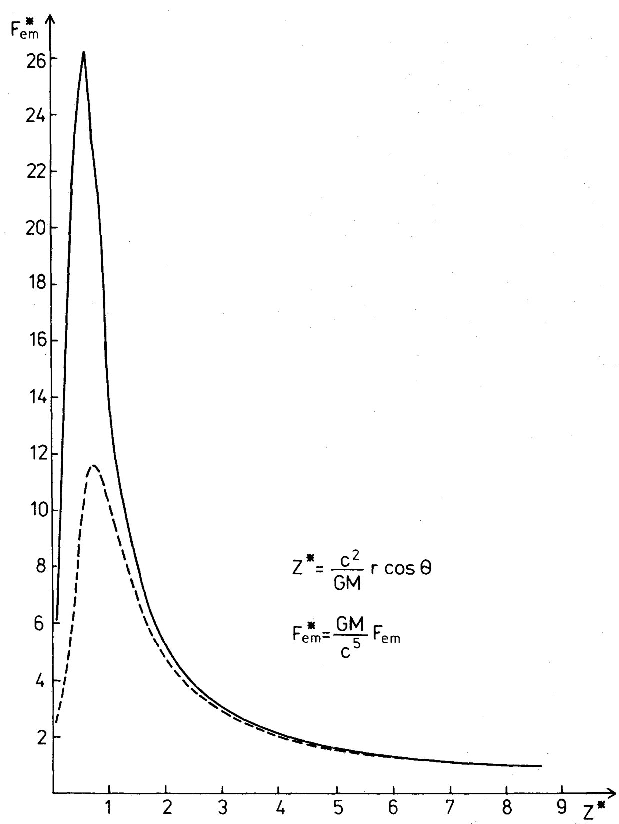

Figure 2. The distribution of surface fluxes in the inner region (solid line – for disc with $r_{in} = r_{mb}$, dotted line – for disc with $r_{in} = r_{ms}$).

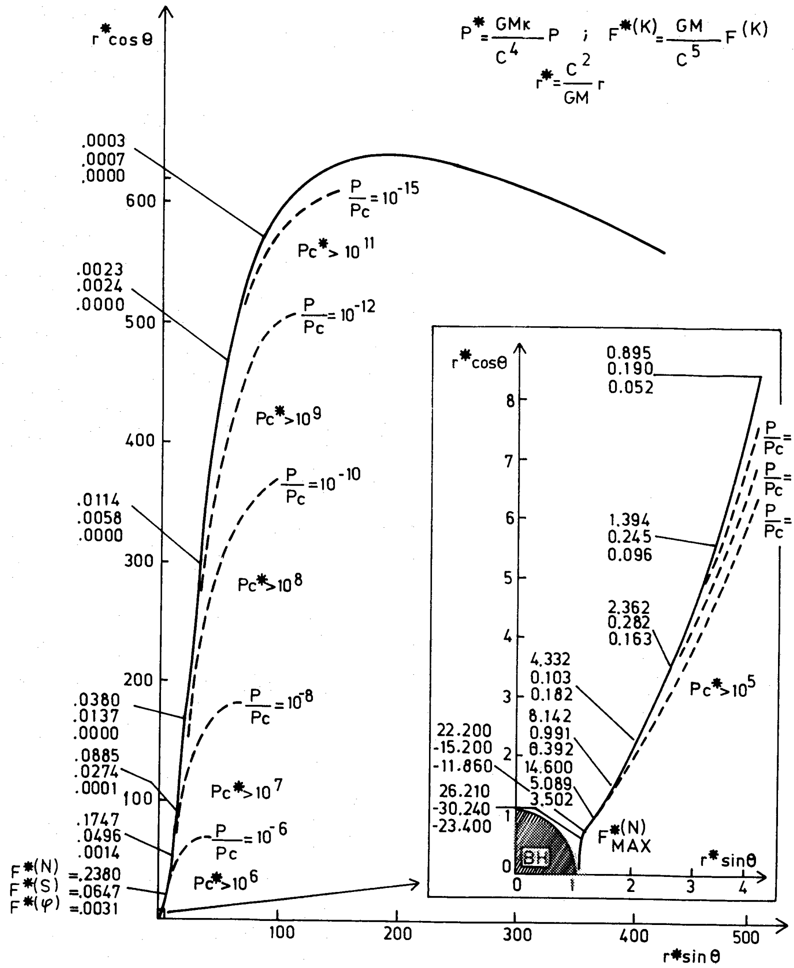

Figure 3. Values of flux components of radiation captured by the $l - \Omega$ ($r_{in} \cong r_{mb}$) disc. Isobars close to the surface of the disc are drawn, and critical values of central pressure required to balance external pressure by pressure on these isobars also indicated.

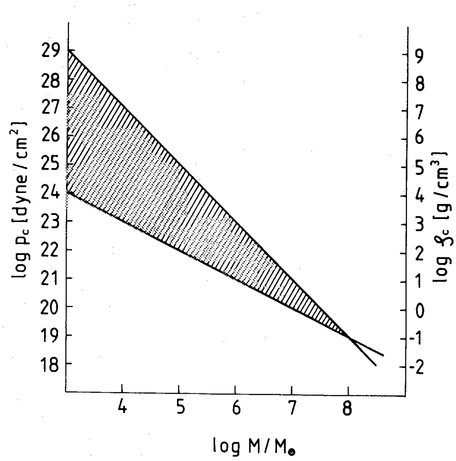

Figure 4. Limits for the central pressure of the disc as a function of the black hole's mass. Upper line corresponds to the requirement that self-gravity effects should be negligible (equation B5); the lower line corresponds to the requirement that the internal pressure be high enough to balance the external pressure (equation B3).

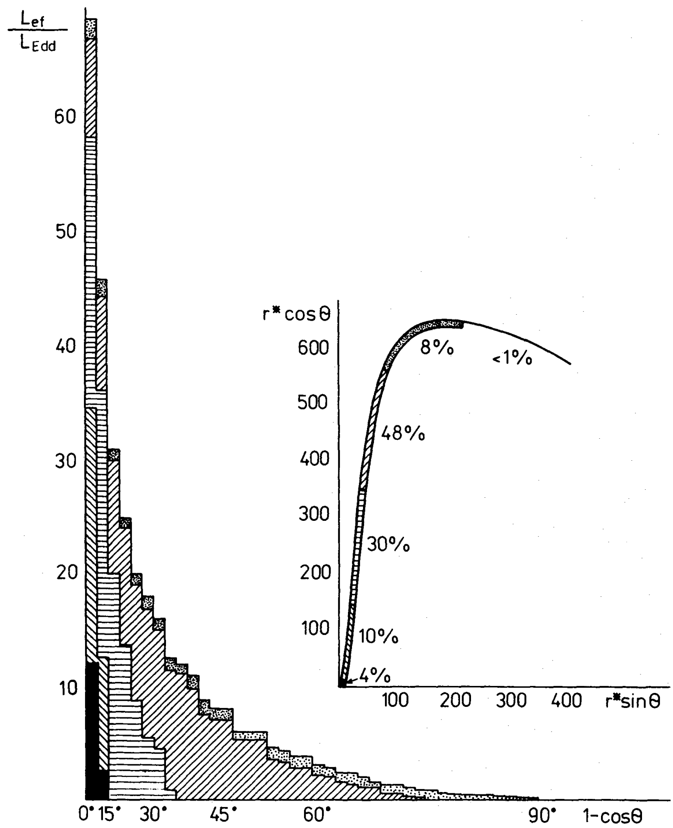

Figure 5. The distributions of the luminosity at infinity as a function of $\cos\theta_{B-L}$. Contributions of different parts of disc to this distribution are also marked.

4 Conclusions

It is impossible to tell now whether the results derived from the JAP model are adequate for quasar nuclei, but it is certain that these results give upper limits for such features as total luminosities and collimation of radiation for accreting discs which are in mechanical equilibrium. Some observational data suggest that the real situation can approach these limits. For example, for quasars with $L > 10^{46}\,\mathrm{erg\,s^{-1}}$, either spherical or thin disc accretion models require masses of the black holes $M_{BH} > 10^8 M_\odot$. If we assume that the quasar phenomenon is an early stage of a galaxy, we might expect observational evidence for such massive black holes in the nuclei of present galaxies. Excepting M87 we have no observational data favouring $M_{BH} > 10^8 M_\odot$. But taking into account results for $r_{out} = 10^3 \times GM_{BH}/c^2$, it is possible to explain $L = 10^{46}\,\mathrm{erg\,s^{-1}}$ by a model with thick accretion disc orbiting black hole of mass only $10^6 M_\odot$ (if we assume a favourable orientation of object relative to us so that $L_{ef}/L_{Edd} \sim 70$ from Fig. 5). The main difficulty of such discs is connected with very high accretion rates required when r_{in} is close to r_{mb} (implying a small efficiency). For an '$l - \Omega$' disc with $r_{out} = 10^3 \times GM_{BH}/c^2$ we have $\dot{M} \sim 10^{-5} M_{BH}\,\mathrm{yr^{-1}}$ (see Table 1); therefore the hole's mass increases by a factor 10 during a period of only $\Delta t = 10^5\,\mathrm{yr}$. For '$\Omega - l - \Omega$' discs the situation radically improves; in such discs with the same r_{out} as above $\dot{M} \sim 8 \times 10^{-8} M_{BH}\,\mathrm{yr^{-1}}$.

As Rees (1980) remarked, very thick discs can also be realized in physical conditions different from those assumed here. The electron and ion temperatures may be different, the pressure being dominated by very hot ions. These conditions can exist, when the time-scale of cooling of ions is longer than the time-scale for inflow of the matter. Such a model would permit a thick disc with very low luminosity in galactic nuclei containing massive black holes, such as M87. Electron–positron plasma, perhaps produced by some exotic process near the hole (Blandford & Znajek 1977; Eilek 1980) and collimated in the funnels, could give observed jets.

Acknowledgments

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References

Abramowicz, M. A., 1974. *Acta astr.*, **24**, 45.
Abramowicz, M. A., Calvani, M. & Nobili, L., 1980. *Astrophys. J.*, **242**, 772.
Abramowicz, M. A., Jaroszyński, M. & Sikora, M., 1978. *Astr. Astrophys.*, **63**, 221.
Blandford, R. D. & Znajek, R. L., 1977. *Mon. Not. R. astr. Soc.*, **179**, 433.
Cunningham, C. T. & Bardeen, J. M., 1973. *Astrophys. J.*, **183**, 237.
Eilek, J. A., 1980. *Astrophys. J.*, **236**, 664.
Jaroszyński, M., Abramowicz, M. A. & Paczyński, B., 1980. *Acta astr.*, **30**, 1.
Kozlowski, M., Jaroszyński, M. & Abramowicz, M. A., 1978. *Astr. Astrophys.*, **63**, 209.
Lynden-Bell, D., 1978. *Phys. Scripta*, **17**, 185.
Misner, Ch. W., Thorne, K. S. & Wheeler, J. A., 1973. In *Gravitation*, Freeman, San Francisco.
Paczyński, B. & Wiita, P., 1980. *Astr. Astrophys.*, **88**, 23.
Rees, M. J., 1980. In *Extragalactic X-Ray Astronomy*, eds Giacconi, R. & Setti, G., Reidel, Holland, p. 339.
Thorne, K. S., 1974. *Astrophys. J.*, **191**, 507.

Appendix A

A.1 NUMERICAL METHOD OF CALCULATING B_k^l

Values of B_k^l can be computed by the following procedure:

1. Establish position of the observer in the kth disc ring and divide his hemisphere into a finite number of small solid angles,

$$\delta\,\Omega_j\ (\sum_j\ \delta\,\Omega_j = 2\pi),$$

with a quasi-isotropic distribution on the hemisphere.

2. For each $\mathbf{n}_j$ chosen as a central direction in the $\delta\,\Omega_j$ calculate constants of the motion, E, Φ, Q, and integrate backwords equation (11) in order to establish whether the observer looking in $\mathbf{n}_j$ direction sees the disc or not.

3. If he does, calculate $g_j = \nu_{\text{obs}j}/\nu_{\text{cm}j}$ and therefore $(1/\pi)g_j^4 n_j^{(\text{N})}\delta\,\Omega_j$, and add these values to the contents of the 'memory-boxes' $\boxed{B_k^l}$ with l depending on which r-rings are observed in the directions $\mathbf{n}_j$.

4. Repeating the above calculations for the observers located in different r-rings obtain the values of all boxes,

$$\left(\frac{1}{\pi}\sum_j g_j^4 n_j^{(\text{N})}\,\delta\,\Omega_j\right)_k^l,$$

which approximate real values of the integrals

$$B_k^l = \frac{1}{\pi}\int_{\Omega(k,l)} g^4 n^{(N)} d\Omega$$

with an error depending on how dense a division of the hemisphere was used. In my calculations I used $\langle\delta\Omega_j\rangle \approx 2\pi/2200$ and many numerical tests preceding my final computations confirm that the error made is less than 4 per cent.

A.2 DERIVATION OF THE FORMULA FOR C_i^k

Locally measured energy, ϵ, carried by photons from the element of the disc per unit of the locally measured time, τ, and per unit of the locally measured surface of the disc element, δS, is:

$$\frac{\delta\mathscr{E}}{\delta S\delta\tau} \equiv F_e = \int_{2\pi} I_e n^{(N)} d\Omega = I_e\pi \quad \text{(isotropic emission).} \tag{A.1}$$

In the case when we calculate the 'energy at infinity', E, per unit of the coordinate time, t, the quantity

$$I_e \equiv \frac{\delta\mathscr{E}}{\delta\tau\delta S\delta\Omega}$$

must be replaced by

$$\frac{\delta E}{\delta t\delta S\delta\Omega} = \frac{\delta\mathscr{E} g_\infty}{(\delta\tau/g_\infty)\delta S\delta\Omega} = g_\infty^2 I_e,$$

where

$$g_\infty = \frac{\nu_\infty}{\nu_{em}},$$

and hence

$$\frac{\delta E}{\delta S\delta t} = I_e\int_{2\pi} g_\infty^2 n^{(N)} d\Omega. \tag{A.2}$$

If we define $\Omega(i, k)$ as the solid angle, measured in the rest frame of the emitter located on the kth disc ring, into which the photons reaching infinity in the ith segment are emitted, then the ratio of the energy carried by photons into the ith segment to the energy carried by all photons is, according to equation (A.2)

$$C_i^k = \frac{\int_{\Omega(i,k)} g_\infty^2 n^{(N)} d\Omega}{\int_{2\pi} g_\infty^2 n^{(N)} d\Omega}. \tag{A.3}$$

A.3 NUMERICAL METHOD OF CALCULATING C_i^k

Values of C_i^k are computed in the following way:

1. Establish the position of the emitter on the kth disc ring and divide his hemisphere into small solid angles, $\delta\,\Omega_j$.
2. For each $\mathbf{n}_j$ calculate $g^2_{\infty j}\, n_j^{(\mathrm{N})}\, \delta\,\Omega_j$ and integrate forwards equation (11) in order to see which photons are reaching infinity.
3. For j connected with photons reaching infinity add values $g^2_{\infty j}\, n_j^{(\mathrm{N})}\, \delta\,\Omega_j$ to the contents of the box $\boxed{C_i^k}$ with i depending on which θ-segments are reached by photons. Simultaneously sum values $g^2_{\infty j}\, n_j^{(\mathrm{N})}\, \delta\,\Omega_j$ in box $\boxed{C^k}$ for all directions and divide the final contents of $\boxed{C_i^k}$ by the value of $\boxed{C^k}$.
4. Repeating the above procedure for all values of k obtain the values

$$\frac{\left(\sum_j g^2_{\infty j}\, n_j^{(\mathrm{N})}\, \delta\,\Omega_j\right)_i^k}{\left(\sum_j g^2_{\infty j}\, n_j^{(\mathrm{N})}\, \delta\,\Omega_j\right)^k}$$

which approximate real values of

$$C_i^k = \frac{\int_{\Omega\,(i,k)} g^2_\infty\, n^{(\mathrm{N})}\, d\Omega}{\int_{2\pi} g^2_\infty\, n^{(\mathrm{N})}\, d\Omega}.$$

Appendix B*

In general, JAP theory does not require any assumptions about physical conditions in the interior of the disc. However, as radiation in the funnels has a very high density, it is necessary to assume something about the interior in order to study the resistance of the disc to external pressure. I shall assume that the disc is barotropic and pressure is dominated by its radiation component, i.e. $p = K\rho^{4/3}$. In this case, thanks to the theorem about von Zeipel cylinders (Abramowicz 1974) I can extrapolate the distribution of the angular momentum from the surface of the disc to its interior. Therefore by using equation (1) I am able to construct isobars in the disc, which coincide with equipotential surfaces $W(r, \theta) = \text{const}$, where potential W is defined by the equality $\nabla W = \mathbf{g}_{\mathrm{ef}}$. Integrating the Euler equation, $\nabla p/(p + \rho c^2) = \mathbf{g}_{\mathrm{ef}}$, with the equation of state as assumed above I obtain the following relations:

$$p/\rho c^2 = \exp\left[(W_0 - W)/(4c^2)\right] - 1 \equiv \delta_W$$

and

$$p = c^8\, \delta_W^4 K^{-3}, \tag{B.1}$$

where W_0 is the potential on the surface of the disc. From the above relations one can find the central pressure, $p_{\mathrm{cen}} = p(r_{\mathrm{cen}})$,† required for p to be equal to the external pressure at some point in the disc near the surface (see Fig. 3), i.e.

$$p_{\mathrm{con}} = (\delta_{W\mathrm{cen}}/\delta_W)^4 p = (\delta_{W\mathrm{cen}}/\delta_W)^4\, (F_{\mathrm{abs}}^{(\mathrm{N})}/c). \tag{B.2}$$

* In this Appendix I use physical units.

† r_{cen} is defined in Fig. 1; p has its maximum value at this point.

Results of calculations made according to the above equations show that the disc is most susceptible to destruction by external pressure in its highest region (see Section 3 and Fig. 3). So, requiring the equality of the external pressure in this region and the internal pressure corresponding to the isobar relatively close to the disc surface in its highest parts, we can estimate the minimal central pressure necessary to ensure that most of the disc is not destroyed. From Fig. 3 we see that the isobar with $p = p_{\text{cen}}\, 10^{-15}$ is adequate for this purpose. As the pressure of external radiation in the highest parts of the funnel is of order $10^{-4} c^4/GM_{\text{BH}}$ we obtain following condition for the central pressure (the lower solid line in Fig. 4)

$$p_{\text{cen}} > 10^{15} p \approx 10^{11} c^4/GM_{\text{BH}} \approx 10^{27} (M_{\text{BH}}/M_\odot)^{-1} \text{ dyn cm}^{-2}. \tag{B.3}$$

An upper limit for p_{cen} can be obtained from the requirement that self-gravity effects should be negligible. Fortunately, for very thick discs a sufficiently good estimate of the disc mass can be obtained by assuming that we have a point source of the gravity potential and a spherical object not contributing to the potential, which is empty below r_{cen}. Then we get:

$$\frac{\rho}{\rho_{\text{cen}}} = \left(\frac{r_{\text{cen}}}{r}\right)^3$$

and

$$M_{\text{disc}} \sim 4\pi \int_{r_{\text{cen}}}^{r_{\text{out}}} \rho r^2\, dr = 4\pi \rho_{\text{cen}} r_{\text{cen}}^3 \ln \frac{r_{\text{out}}}{r_{\text{cen}}}.$$

Now from the assumption that $M_{\text{disc}}/M_{\text{BH}} < 1$ we obtain:

$$\rho_{\text{cen}} < \frac{M_{\text{BH}}}{4/3\pi r_{\text{cen}}^3} \frac{1}{3 \ln (r_{\text{out}}/r_{\text{cen}})} \approx 10^{17} \frac{1}{r_{\text{cen}}^{*3}} \left(\frac{M_{\text{BH}}}{M_\odot}\right)^{-2} \frac{1}{3 \ln r_{\text{out}}/r_{\text{cen}}} \text{ g cm}^{-3}, \tag{B.4}$$

where

$$r_{\text{cen}}^* = r_{\text{cen}}/(GM_{\text{BH}}/c^2).$$

For the case of the discs calculated in this work we have (the upper solid line in Fig. 4)

$$\rho_{\text{cen}} < 10^{15} (M_{\text{BH}}/M_\odot)^{-2} \text{ g cm}^{-3}$$

and

$$p_{\text{cen}} = \delta w_{\text{cen}} \rho_{\text{cen}} c^2 < 10^{35} (M_{\text{BH}}/M_\odot)^{-2} \text{ dyn cm}^{-2}. \tag{B.5}$$