Introduction to Magnetohydrodynamics

part B of "Selected Topics in High Energy Astrophysics"

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Part I

HYDRODYNAMICS

We approximate the cosmic plasma as a continuous medium, treating it as a **fluid that is electrically neutral**. The first assumption, considering the collisionless nature of the plasma, necessitates the presence of a magnetic field to confine particles within the system. The second assumption entails averaging over time scales and spatial scales larger than the inverse of the plasma frequency and the Debye length, respectively.

General references for MHD tutorials and textbooks

- Kulsrud, R. M. (2005). ***Plasma Physics for Astrophysics*.**
- Ogilvie, G. I. (2016).**Lecture notes: Astrophysical fluid dynamics.** *arXiv e-prints*, Article arXiv:1604.03835, arXiv:1604.03835. https://doi.org/10.48550/arXiv.1604.03835
- Spruit, H. C. (2013).**Essential Magnetohydrodynamics for Astrophysics.** *arXiv e-prints*, Article arXiv:1301.5572, arXiv:1301.5572. https://doi.org/10.48550/arXiv.1301.5572

Preliminaries

Basic definitions

Let us define macroscopic scalar parameters of the jet fluid:

- **number density** n,
- **pressure** p,
- **internal energy density** (including the rest-mass energy density) ϵ,
- **enthalpy** $w = \epsilon + p$.

All of these quantities are measured in the rest frame of the fluid, i.e. per **proper** unite volume, and therefore should be called “the proper number density”, “the proper pressure”, etc. (“primitive variables”). Note that the **proper specific internal energy** ε is defined as

$$\epsilon = mnc^2 + \varepsilon \quad \text{so that} \quad w = mnc^2 + \varepsilon + p \tag{1}$$

where **the proper (rest) mass density** is mn for a mass of a fluid particle m.

The four-velocity of the fluid is $u^\mu = (\Gamma, \Gamma\, \beta^k)$, where $\vec{v} \equiv (\beta^k c)$ is the bulk 3-velocity, $\Gamma \equiv (1 - \beta^2)^{-1/2}$ is the bulk Lorentz factor, and the indices $\mu = 0, 1, 2, 3$ and $k = 1, 2, 3$. In the fluid rest frame one has $u'^\mu = (1, 0, 0, 0)$.

Preliminaries

Stress-energy tensor

For an **ideal fluid** (no energy dissipation, etc.), and in the absence of external forces, the **fluid stress-energy tensor** is diagonal in the fluid rest frame, namely

$$\mathcal{T}^{\mu\nu} = w\, u^\mu u^\nu - p\, g^{\mu\nu} \quad , \tag{2}$$

where $g^{\mu\nu}$ is the metric tensor of the **Minkowski spacetime**, with the $(+---)$ signature adopted here.

The **particle flux four-vector** is simply

$$\mathcal{D}^\mu = n\, u^\mu \quad . \tag{3}$$

The following individual components of the stress-energy tensor and particle flux vector can be identified with, respectively,

- **the total energy density** $\mathcal{T}^{00} = w\,\Gamma^2 - p = \left(\epsilon + p\beta^2\right)\,\Gamma^2$
- **the energy flux density** $\mathcal{T}^{0k} = w\,\Gamma^2\,\beta^k$
- **the momentum flux density** $\mathcal{T}^{ik} = w\,\Gamma^2\,\beta^i\beta^k + p\,\delta^{ik}$
- **the particle number density** $\mathcal{D}^0 = n\,u^0 = n\,\Gamma$
- **the particle flux density** $\mathcal{D}^k = n\,u^k = n\,\Gamma\,\beta^k$

The local conservation laws in relativistic ideal hydrodynamics are obtained from vanishing divergence of the stress-energy tensor of the fluid,

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 0 \tag{4}$$

(energy-momentum conservation), and of the particle flux,

$$\nabla_\mu \mathcal{D}^{\mu} = 0 \tag{5}$$

(particle conservation), where $\nabla_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\partial_t, \nabla_k)$ is the covariant differential operator.

Ideal fluid

In Cartesian coordinates

The conservation laws $\nabla_\mu \mathcal{D}^\mu$ and $\nabla_\mu \mathcal{T}^{\mu\nu} = 0$, in Cartesian coordinates, become

$$\partial_t(n\Gamma) + \partial_i(n\Gamma\beta^i c) = 0 \tag{6}$$

$$\partial_t(w\Gamma^2 - p) + \partial_i(w\Gamma^2\beta^i c) = 0 \tag{7}$$

$$\partial_t(w\Gamma^2\beta^k) + \partial_i(w\Gamma^2\beta^i\beta^k c + pc\delta^{ik}) = 0 \tag{8}$$

Let us now define the following quantities **measured in the laboratory frame**:

- **the rest mass density** $\rho \equiv m\mathcal{D}^0 = mn\Gamma$
- **the total energy density** $U \equiv \mathcal{T}^{00} = w\Gamma^2 - p$
- **the momentum density vector** $\vec{P} \equiv \mathcal{T}^{0k}/c = w\Gamma^2\vec{\beta}/c$

(note that $\rho \neq mn$). With such, the conservations laws become

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{9}$$

$$\partial_t U + \partial_i(Uv^i + pv^i) = 0 \tag{10}$$

$$\partial_t P^k + \partial_i(P^k v^i + p\delta^{ik}) = 0 \tag{11}$$

IDEAL FLUID

NEWTONIAN QUANTITIES

In the non-relativistic limit, it is convinient to subtract the rest-mass density from the total energy density, $\tilde{U} \equiv U - \rho c^2$, noting that this will not affect the energy conservation law, namely $\partial_t \tilde{U} + \partial_i(\tilde{U} v^i + p v^i) = 0$. With such, **Newtonian counterparts** for the quantities ρ, $\tilde{U}$, and $\vec{P}$, can be found by series expansions for non-relativistic bulk velocity $\beta \to 0$ (and so $\Gamma \simeq 1 + \frac{1}{2}\beta^2$ and $\Gamma^2 \simeq 1 + \beta^2$), assuming moreover "cold plasma" $mnc^2 \gg \varepsilon + p$ in the momentum equation, namely

$$\rho = mn\Gamma \;\; \to \;\; mn\,(1 + \frac{1}{2}\beta^2) + \mathcal{O}(\beta^4) \;\; \simeq mn \tag{12}$$

$$\begin{aligned} \tilde{U} &= \rho c^2(\Gamma - 1) + \varepsilon\Gamma^2 + p(\Gamma^2 - 1) \;\; \to \;\; \frac{1}{2}\rho c^2\beta^2 + \varepsilon(1 + \beta^2) + p\beta^2 + \mathcal{O}(\beta^4) \\ &\simeq \frac{1}{2}\rho v^2 + \varepsilon \end{aligned} \tag{13}$$

$$\begin{aligned} \vec{P} &= (mnc^2 + \varepsilon + p)\Gamma^2\vec{\beta}/c \simeq mnc^2\Gamma^2\vec{\beta}/c = \rho c^2\Gamma\vec{\beta}/c \;\; \to \;\; \rho c^2\vec{\beta}/c + \mathcal{O}(\beta^3) \\ &\simeq \rho\vec{v} \qquad \textit{(in the momentum equation only!)} \end{aligned} \tag{14}$$

All in all, the set of equations describing **non-relativistic ideal hydrodynamics**, can therefore be written in the form of the **Euler conservation laws**:

$$\textbf{mass} \qquad \partial_t \rho + \partial_i(\rho v^i) = 0 \tag{15}$$

$$\textbf{energy} \qquad \partial_t(\frac{1}{2}\rho v^2 + \varepsilon) + \partial_i(\frac{1}{2}\rho v^2 + \varepsilon + p)v^i = 0 \tag{16}$$

$$\textbf{momentum} \qquad \partial_t(\rho v^k) + \partial_i(\rho v^k v^i + p\delta^{ik}) = 0 \tag{17}$$

Note the general form of these equation "$\partial_t \, \mathrm{stuff} + \vec{\nabla} \cdot \mathrm{flux\ of\ stuff} = 0$", which may be therefore expressed in the integral forms, by integrating over volume $\mathcal{V}$ and using the **Gauss theorem**

$$\int \left(\vec{\nabla} \cdot \vec{F}\right) \, d\mathcal{V} = \int_{\partial\mathcal{V}} \vec{F} \cdot d\vec{\mathcal{S}} \tag{18}$$

where the volume's surface $\partial\mathcal{V} \equiv \mathcal{S}$ with the outward-pointing normal unit vector $\hat{n}$ for each differential surface, $d\vec{\mathcal{S}} = \hat{n} \, d\mathcal{S}$.

Now, let us re-write the mass conservation law, $\partial_t \rho + \partial_i(\rho v^i) = 0$, as

$$(\partial_t + v^i \partial_i)\rho = -\rho\, \partial_i v^i \tag{19}$$

and denote the **convective derivative** (‘comoving derivative’, ‘material derivative’, ‘substantial derivative’,...) as

$$D_t \equiv \partial_t + v^i \partial_i \tag{20}$$

which is measuring the changes of a quantity as it follows a fluid flow:

$$\begin{aligned}
\Delta f &= f(t+\Delta t, \vec{x} + \vec{v}\Delta t) - f(t, \vec{x}) \\
&\simeq \left[f(t,\vec{x}) + \Delta t\ \partial_t f(t,\vec{x}) + \Delta t\, \vec{v} \cdot \vec{\nabla} f(t,\vec{x})\right] - f(t,\vec{x}) \\
&= \Delta t \left(\partial_t + \vec{v} \cdot \vec{\nabla}\right) f(t,\vec{x}) \qquad (21)\\
\rightarrow \quad d_t f &\equiv \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \left(\partial_t + \vec{v} \cdot \vec{\nabla}\right) f(t,\vec{x}) \qquad (22)
\end{aligned}$$

Accordingly, all the combining and re-arranged Euler conservation laws may be written in the compact **Lagrangian form**

$$\textbf{mass} \qquad D_t\,\rho = -\rho\,\vec{\nabla}\cdot\vec{v} \tag{23}$$

$$\textbf{energy} \qquad D_t\left(\frac{\varepsilon}{\rho}\right) = -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} \tag{24}$$

$$\textbf{momentum} \qquad D_t\,\vec{v} = -\frac{1}{\rho}\,\vec{\nabla}p \tag{25}$$

Continuity equation

Various formulations

Let's look at various form form of the law for the conservation of fluid mass, aka the continuity equation:

$$\textbf{Euler} \qquad \partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{26}$$

$$\textbf{Gauss} \qquad \partial_t \int d\mathcal{V}\,\rho = \int_{\partial\mathcal{V}} d\vec{\mathcal{S}}\cdot(\rho\vec{v}) \tag{27}$$

$$\textbf{Lagrange} \qquad \frac{1}{\rho}D_t\rho = -\vec{\nabla}\cdot\vec{v} \tag{28}$$

- the first one says that the time derivative of a mass density at a given place is balanced by the divergence of the mass density flux
- the second one is telling us that the variation of mass in the volume $\mathcal{V}$ must be entirely due to the -in or -outflow of mass through the volume's surface $\partial\mathcal{V} \equiv \mathcal{S}$
- the third one is telling us that the relative change in the element of a fluid *along* the flow, $\frac{D_t\rho}{\rho}$, is equal to the fluid compression $-\vec{\nabla}\cdot\vec{v}$ ("converging/diverging flow").

Energy equation

Fundamental thermodynamic relation

Let us recall the **fundamental thermodynamic relation** (i.e., mathematical summation of the first and second law of thermodynamics):

$$dU = T\,dS - p\,d\mathcal{V} \tag{29}$$

where T is the **temperature**, $U = \varepsilon\mathcal{V}$ is the internal energy of a fluid, and S is the **fluid entropy**. Noting that $\rho = M/\mathcal{V}$, where M is the total mass within the volume $\mathcal{V}$, one therefore has $U = \varepsilon M/\rho$, and hence, assuming M is constant,

$$d\left(\frac{\varepsilon}{\rho}\right) = T\,ds - p\,d\left(\frac{1}{\rho}\right) \tag{30}$$

where $s \equiv S/M$ is the **specific entropy**, i.e. fluid entropy per unit mass.

Now, let us note that $d(1/\rho) = -\rho^{-2} d\rho$, replace the total derivative with the convective derivative, $d \to D_t$, and recall the continuity equation in the Lagrangian form $D_t\rho = -\rho\,\vec{\nabla} \cdot \vec{v}$; with all of such we arrive at

$$D_t \left(\frac{\varepsilon}{\rho} \right) = T\, D_t s - \frac{p}{\rho}\, \vec{\nabla} \cdot \vec{v} \tag{31}$$

This, when compared with the energy equation in the Lagrangian form, implies that

$$D_t\, s = 0 \tag{32}$$

i.e., that the **specific entropy is conserved along the flow** (that is, following a volume element along the flow) or, in other words, that the ideal fluid is adiabatic.

Energy equation

Equation of state

Let us introduce the **equation of state** as the relation

$$p = (\hat{\gamma} - 1)\varepsilon \tag{33}$$

where $\hat{\gamma}$ is the adiabatic index, and recall again the energy equation in the Lagrangian form $D_t\left(\frac{\varepsilon}{\rho}\right) = -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v}$. From there it follows that

$$D_t\!\left(\frac{p}{\rho}\right) = -(\hat{\gamma} - 1)\,\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} = (\hat{\gamma} - 1)\,\frac{p}{\rho^2}\,D_t\rho \tag{34}$$

(assuming that $\hat{\gamma}$ is constant along the flow!), where we used again the continuity equation. This is equivalent to

$$D_t \ln p = \hat{\gamma}\, D_t \ln\rho \tag{35}$$

meaning that

$$D_t\left(\frac{p}{\rho^{\hat{\gamma}}}\right) = 0 \tag{36}$$

We have therefore equivalent equations

$$
\begin{aligned}
\textbf{energy conservation} \quad D_t\left(\frac{\varepsilon}{\rho}\right) &= -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} && (37)\\
\textbf{+ thermodynamic relation} \quad D_t\, s &= 0 && (38)\\
\textbf{+ equation of state} \quad D_t\left(\frac{p}{\rho^{\hat{\gamma}}}\right) &= 0 && (39)
\end{aligned}
$$

Note that the fundamental thermodynamic relation with the specific entropy conserved, $ds = 0$, reads as

$$d\left(\frac{\varepsilon}{\rho}\right) = -p\, d\left(\frac{1}{\rho}\right) \tag{40}$$

which, for the equation of state $p = (\hat{\gamma} - 1)\varepsilon$, may be re-arranged as

$$d \ln p = \hat{\gamma}\, d \ln \rho \tag{41}$$

meaning

$$p = K\, \rho^{\hat{\gamma}} \tag{42}$$

where K is the integration constant (*for an adiabatic process*). The above is called the **polytropic equation of state**. What is therefore a meaning of the statement $D_t(p/\rho^{\hat{\gamma}}) \equiv D_t K = 0$? And how does it relate to the specific entropy conservation $D_t s = 0$?

Energy equation

Forms of entropy

Recall that $\varepsilon = p/(\hat{\gamma} - 1) = K\rho^{\hat{\gamma}}/(\hat{\gamma} - 1)$, so that, *assuming K is a variable*,

$$d \ln \varepsilon = \hat{\gamma} \; d \ln \rho + d \ln K \tag{43}$$

Moreover, since $p = (\rho/m)kT$, the fluid temperature is

$$T = \frac{m}{k}\frac{p}{\rho} \tag{44}$$

Using these, the fundamental thermodynamic relation $d(\varepsilon/\rho) = T\,ds - p\,d(1/\rho)$ can be re-written as

$$\frac{m}{k}\; ds = \frac{1}{\hat{\gamma} - 1}\; d \ln K \tag{45}$$

meaning that K is, in fact, **a form of entropy**,

$$s = s_0 + \frac{k}{m\,(\hat{\gamma} - 1)}\; \ln K \tag{46}$$

and hence the conditions $D_t\, s = 0$ and $D_t\,(p/\rho^{\hat{\gamma}}) = 0$ are indeed equivalent.

Finally, let us look again at the momentum conservation equation in the Lagrangian form,

$$\rho\, D_t\, \vec{v} = -\vec{\nabla} p \tag{47}$$

which is clearly the **equation of motion** (recall the Newton's $m\, d_t \vec{v} = \vec{F}$), or the **Navier-Stokes equation** for an **ideal inviscid fluid**. It says that the element of a fluid will experience acceleration along the flow due to a force being the pressure gradient, $\vec{F} = -\vec{\nabla} p$. Note that any other force, such as gravity, or a Lorenz force, can therefore be incorporated as an additional term on the right-hand side of this equation.

Part II

MAGNETOHYDRODYNAMICS

Let us define the **electromagnetic potential**

$$\mathcal{A}^{\mu} = (\varphi, \vec{A}) \tag{48}$$

such that the magnetic field intensity $\vec{B} = \vec{\nabla} \times \vec{A}$ and the electric field $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t \vec{A}$. The **electromagnetic field tensor** is

$$\mathcal{F}_{\mu\nu} = \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu} \tag{49}$$

so that $\mathcal{F}^{0k} = -E^k$ and $\mathcal{F}^{ik} = -\varepsilon_{ikm}B^m$. Note that $\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu}$, and also the invariants $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = 2\,(B^2 - E^2) = inv$, along with $\varepsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} = -8\,(\vec{E}\cdot\vec{B}) = inv$.

We moreover introduce the relativistic **four-vector electromagnetic current** as

$$\mathcal{J}^{\mu} = (c\,Q, \vec{j}) \tag{50}$$

where Q is the electric charge density, and $\vec{j}$ is the electric current density vector.

Maxwell's equations can now be formulated as

$$\nabla_\nu \mathcal{F}^{\mu\nu} = -\frac{4\pi}{c}\mathcal{J}^\mu \tag{51}$$

$$\epsilon^{\alpha\beta\mu\nu}\nabla_\beta \mathcal{F}_{\mu\nu} = 0 \tag{52}$$

In the Cartesian coordinates they obtain the familiar forms:

$$\textbf{Amper} \quad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\vec{E} \tag{53}$$

$$\textbf{Faraday} \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B} \tag{54}$$

$$\textbf{Gauss} \quad \vec{\nabla}\cdot\vec{B} = 0 \tag{55}$$

$$\textbf{Poisson} \quad \vec{\nabla}\cdot\vec{E} = 4\pi Q \tag{56}$$

(here we use strictly **Gauss units!!!**).

The **stress-energy tensor of the EM field** is defined as

$$\mathcal{T}^{\mu\nu}_{\mathrm{EM}} = -\frac{1}{4\pi}\,\mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}{}_{\alpha} + \frac{1}{16\pi}\,g^{\mu\nu}\,\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} \quad , \tag{57}$$

The following individual components of this stress-energy tensor can be identified with

- **the EM field energy density** $\mathcal{T}^{00}_{\mathrm{EM}} \equiv U_{\mathrm{EM}} = \frac{1}{8\pi}\,(E^2 + B^2)$
- **the EM field energy density (Poynting) flux** $c\mathcal{T}^{i0}_{\mathrm{EM}} \equiv P^{i}_{\mathrm{EM}} = \frac{c}{4\pi}\,(\vec{E}\times\vec{B})^{i}$
- **the EM field momentum flux density** $\mathcal{T}^{ik}_{\mathrm{EM}} \equiv \Pi^{ik}_{\mathrm{EM}} = -\frac{1}{4\pi}\left(E^iE^k + B^iB^k\right) + \frac{1}{8\pi}\left(E^2 + B^2\right)\,\delta^{ik}$

Note the two components in the momentum flux density (**“Maxwell stress”**) tensor, corresponding to the tension and pressure of the field lines, respectively. Also, $\mathcal{T}^{\mu\nu}_{\mathrm{EM}} = \mathcal{T}^{\nu\mu}_{\mathrm{EM}}$.

Recall the **Lorentz transformations** of four-vectors and tensors:

$$\mathcal{J}'^{\mu} = \Lambda^{\mu}{}_{\alpha}\, \mathcal{J}^{\alpha} \quad , \quad \mathcal{F}'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\, \mathcal{F}^{\alpha\beta} \tag{58}$$

where the Lorentz transformation matrix

$$\Lambda^{\mu}{}_{\alpha} = \begin{bmatrix} \Gamma & -\Gamma\beta_x & -\Gamma\beta_y & -\Gamma\beta_z \\ -\Gamma\beta_x & 1+\frac{\Gamma-1}{\beta^2}\beta_x^2 & \frac{\Gamma-1}{\beta^2}\beta_x\beta_y & \frac{\Gamma-1}{\beta^2}\beta_x\beta_z \\ -\Gamma\beta_y & \frac{\Gamma-1}{\beta^2}\beta_x\beta_y & 1+\frac{\Gamma-1}{\beta^2}\beta_y^2 & \frac{\Gamma-1}{\beta^2}\beta_y\beta_z \\ -\Gamma\beta_z & \frac{\Gamma-1}{\beta^2}\beta_x\beta_z & \frac{\Gamma-1}{\beta^2}\beta_y\beta_z & 1+\frac{\Gamma-1}{\beta^2}\beta_z^2 \end{bmatrix} \tag{59}$$

Keep in mind that

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{60}$$

It therefore follows that the relativistic transformation of the electric and magnetic field components are

$$\vec{E}' = \Gamma \left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{E}\right) \tag{61}$$

$$\vec{B}' = \Gamma \left(\vec{B} - \vec{\beta} \times \vec{E}\right) - \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{B}\right) \tag{62}$$

Note that **Lorentz transformations effectively “mix” the electric and magnetic field components**.

As for the electric charge density and currents, we have

$$cQ' = \Gamma\, cQ - \Gamma \left(\vec{\beta} \cdot \vec{j}\right) \tag{63}$$

$$\vec{j}' = \vec{j} - \Gamma\, cQ\, \vec{\beta} + \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{j}\right) \tag{64}$$

According to the **classical Ohm's law**, in the rest frame of a fluid the electric current is parallel to the electric field,

$$\vec{j}' = \sigma\, \vec{E}' \tag{65}$$

where σ is the fluid **conductivity**. Taking now the transformation of the EM field and currents in the non-relativistic regime, i.e., ignoring terms $\mathcal{O}(\beta^2)$ or higher, namely

$$\vec{E}' \simeq \vec{E} + \vec{\beta} \times \vec{B} \quad , \quad \vec{B}' \simeq \vec{B} - \vec{\beta} \times \vec{E} \quad , \quad \vec{j}' \simeq \vec{j} - \vec{v} Q \tag{66}$$

one therefore obtains

$$\frac{1}{\sigma} \left(\vec{j} - \vec{v} Q \right) \simeq \vec{E} + \vec{\beta} \times \vec{B} \tag{67}$$

meaning $\vec{E} \simeq -\vec{\beta} \times \vec{B}$ in the perfect conductivity limit, $\sigma^{-1} \to 0$.

The essential statement here is, in fact, that **the electric field must vanish in the fluid rest frame, if only the conductivity is infinite**,

$$\vec{E}' = 0 \quad \text{if} \quad \sigma^{-1} \to 0 \tag{68}$$

because in this limit charge carriers immediately rearrange to cancel all the rest-frame electric fields.

Assuming therefore the perfect conductivity limit, $\sigma^{-1} \to 0$, the **covariant form of the ideal Ohm's law** is

$$\mathcal{F}^{\mu\nu} u_\nu = 0 \tag{69}$$

This is not a full relativistic generalization of the Ohm's law, but only the covariant form assuring that, in the perfect conductivity limit, electric field is vanishing in the fluid rest frame. Indeed, note that in the fluid rest frame $\mathcal{F}'^{\mu\nu} u'_\nu = (0, \vec{E}')$, while in general $(u^\mu) = (\Gamma, \Gamma\,\beta^k)$, so that for the space components

$$\mathcal{F}^{i\nu} u_\nu = 0 \quad \to \quad \Gamma \left(\vec{E} + \vec{\beta} \times \vec{B}\right) = 0 \tag{70}$$

while the time component gives the consistency condition which is then automatically satisfied, namely

$$\mathcal{F}^{0\nu} u_\nu = 0 \quad \to \quad \Gamma \left(\vec{E} \cdot \vec{\beta}\right) = 0 \tag{71}$$

Note that for an ideal electric field $\vec{E} = -\vec{\beta} \times \vec{B}$, the Poynting flux becomes

$$\vec{P}_{\mathrm{EM}} = \frac{c}{4\pi} \left(\vec{E} \times \vec{B}\right) = \frac{c}{4\pi} \left[\left(\vec{\beta}\, B^2 - \vec{B}\,(\vec{\beta} \cdot \vec{B}\right)\right] \tag{72}$$

INTERLUDE
POYNTING THEOREM

Recall the identity which follows from the Maxwell's equations for given definitions of $\mathcal{T}_{\mathrm{EM}}^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$,

$$\nabla_\mu \mathcal{T}_{\mathrm{EM}}^{\mu\nu} = -\frac{1}{c}\,\mathcal{F}^{\nu\alpha}\,\mathcal{J}_\alpha \tag{73}$$

This relation describes the exchange of energy and momentum between the EM field and a matter, with the matter entering only through the 4-current $\mathcal{J}_\alpha$.

Let's consider first the time component of this identity, and in particular its both sides

$$\nabla_\mu \mathcal{T}_{\mathrm{EM}}^{\mu 0} = \frac{1}{c}\,\partial_t\,U_{\mathrm{EM}} + \vec{\nabla}\cdot\frac{1}{c}\vec{P}_{\mathrm{EM}} \quad \text{and} \quad -\frac{1}{c}\mathcal{F}^{0\alpha}\,\mathcal{J}_\alpha = -\frac{1}{c}\vec{j}\cdot\vec{E} \tag{74}$$

respectively. We have therefore

$$-\partial_t\,U_{\mathrm{EM}} = \vec{\nabla}\cdot\vec{P}_{\mathrm{EM}} + \vec{j}\cdot\vec{E} \tag{75}$$

i.e. the **Poynting theorem**, which can be also expressed in the integral form

$$-d_t\int U_{\mathrm{EM}}\,d\mathcal{V} = \int_{\partial\mathcal{V}}\vec{P}_{\mathrm{EM}}\cdot d\vec{\mathcal{S}} + \int \vec{j}\cdot\vec{E}\,d\mathcal{V} \tag{76}$$

The rate of changes of the EM field energy in a given volume is equal to the EM energy flowing in/out of the volume, minus the EM energy *dissipated* within this volume at the rate $\vec{j}\cdot\vec{E}$.

Interlude

Lorentz force

Now let's consider the space component of the $\nabla_\mu \mathcal{T}^{\mu\nu}_{\mathrm{EM}} = -\frac{1}{c}\,\mathcal{F}^{\nu\alpha}\,\mathcal{J}_\alpha$ identity, for which the both sides are

$$\nabla_\mu \mathcal{T}^{\mu k}_{\mathrm{EM}} = \frac{1}{c^2}\,\partial_t P^k_{\mathrm{EM}} + \nabla_i\,\Pi^{ki}_{\mathrm{EM}} \quad \text{and} \quad -\frac{1}{c}\mathcal{F}^{i\alpha}\,\mathcal{J}_\alpha = -Q\,\vec{E} - \frac{1}{c}\vec{j}\times\vec{B} \tag{77}$$

respectively. From this we obtain the equivalent of the momentum equation for the EM field, namely

$$\frac{1}{c^2}\,\partial_t \vec{P}_{\mathrm{EM}} + \vec{\nabla}\cdot\hat{\Pi}_{\mathrm{EM}} = -\vec{F}_{\mathrm{L}} \tag{78}$$

where the **Lorentz force** density is

$$\vec{F}_{\mathrm{L}} = Q\,\vec{E} + \frac{1}{c}\vec{j}\times\vec{B} \tag{79}$$

This is the **force exerted by the EM field on the fluid!**

The standard covariant formulation of relativistic ideal **magnetohydrodynamics (MHD)**, consists of

$$
\begin{aligned}
\text{total energy-momentum conservation} &\qquad \nabla_\mu \left(\mathcal{T}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{\mathrm{EM}}\right) = 0 && (80)\\
\text{particle number conservation} &\qquad \nabla_\mu \mathcal{D}^\mu = 0 && (81)\\
\text{Maxwell's equations (inhomogeneous)} &\qquad \nabla_\nu \mathcal{F}^{\mu\nu} = -\frac{4\pi}{c}\mathcal{J}^\mu && (82)\\
\text{Maxwell's equations (homogeneous)} &\qquad \epsilon^{\alpha\beta\mu\nu}\nabla_\beta \mathcal{F}_{\mu\nu} = 0 && (83)\\
\text{ideal Ohm's law} &\qquad \mathcal{F}^{\mu\nu} u_\nu = 0 && (84)
\end{aligned}
$$

Note that the charge conservation, $\nabla_\mu \mathcal{J}^\mu = 0$, is not an independent equation in this framework, but it follows automatically from Maxwell's equations, since $\nabla_\mu \nabla_\nu \mathcal{F}^{\mu\nu} \equiv 0$ for $\mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu}$.

MHD equations describe dynamics of a conducting fluid interacting with electromagnetic field.

Orders-of-magnitude analysis of the Faraday law implies

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\,\partial_t \vec{B} \quad \rightarrow \quad \frac{E}{\ell} \sim \frac{B\,v}{c\,\ell} \quad \rightarrow \quad E \sim \frac{v}{c}\,B \tag{85}$$

where ℓ is the characteristic spatial scale, and the dynamical timescale $\tau \sim \ell/v$ for the fluid velocity v. Hence, the **"displacement current"**

$$\frac{1}{c}\partial_t \vec{E} \quad \rightarrow \quad \frac{E\,v}{c\,\ell} \sim \left(\frac{v}{c}\right)^2 \frac{B}{\ell} \tag{86}$$

This implies in particular that, **in the non-relativistic regime** $v/c \ll 1$**, currents dominate the dynamics of the EM field**, since through the Amper's law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \quad \rightarrow \quad \frac{B}{\ell} \sim \frac{j}{c} \quad \gg \quad \left(\frac{v}{c}\right)^2 \frac{B}{\ell} \tag{87}$$

In the non-relativistic regime one can therefore neglect the $\mathcal{O}(\beta^2)$-order displacement current $c^{-1}\partial_t \vec{E}$.

Similarly, orders-of-magnitude analysis of the Poisson law implies

$$\vec{\nabla}\cdot\vec{E} = 4\pi Q \quad \rightarrow \quad \frac{E}{\ell} \sim Q \tag{88}$$

and therefore

$$Q\vec{E} \quad \rightarrow \quad \frac{E^2}{\ell} \sim \left(\frac{v}{c}\right)^2 \frac{B^2}{\ell} \tag{89}$$

This implies in particular that, **in the non-relativistic regime $v/c \ll 1$, currents also dominate the force exerted by the EM field on the fluid**, since

$$\frac{1}{c}\vec{j}\times\vec{B} \quad \rightarrow \quad \frac{j\,B}{c} \sim \frac{B^2}{\ell} \quad \gg \quad \left(\frac{v}{c}\right)^2 \frac{B^2}{\ell} \tag{90}$$

In the non-relativistic regime one can therefore neglect the $\mathcal{O}(\beta^2)$-order term $Q\vec{E}$ in the Lorentz force.

Ideal MHD

Magnetic Induction Equation

Let us therefore simplify Maxwell's equations by ignoring the displacement current, $c^{-1}\partial_t\vec{E} = 0$, as well as by setting charge density to zero, $Q = 0$, both of which assumptions are justified in the non-relativistic regime, as elaborated above. One then has in particular

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\,\vec{j} \quad , \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\,\partial_t\vec{B} \quad , \quad \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{E} = 0 \tag{91}$$

We also recall the non-relativistic Ohm's law $\sigma^{-1}\vec{j} = \vec{E} + \vec{\beta} \times \vec{B}$. By combining the above relations, and noting that $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\vec{\nabla}^2\vec{B}$, one obtains the equation governing an evolution of the magnetic field, aka the **magnetic induction equation**,

$$\partial_t\vec{B} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) + \eta\,\vec{\nabla}^2\vec{B} \tag{92}$$

where the **magnetic diffusivity** is

$$\eta \equiv \frac{c^2}{4\pi\sigma} \tag{93}$$

The first term on the right-hand side of this equation described **advection** of the magnetic field with the fluid, while the second term corresponds to the **diffusion** of the magnetic field out of the system.

Order-of-magnitude analysis of the two terms on the right-hand side of the magnetic induction equation:

$$\vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) \quad \rightarrow \quad \frac{B\,v}{\ell} \equiv \frac{B}{\tau_{\rm adv}} \quad \rightarrow \quad \tau_{\rm adv} = \frac{\ell}{v} \tag{94}$$

$$\eta\,\vec{\nabla}^2 \vec{B} \quad \rightarrow \quad \frac{B\,\eta}{\ell^2} \equiv \frac{B}{\tau_{\rm diff}} \quad \rightarrow \quad \tau_{\rm diff} = \frac{\ell^2}{\eta} \tag{95}$$

where ℓ is the characteristic spatial scale of the system. The **magnetic Reynolds number** is a ratio of the two corresponding timescales for the field advection and diffusion,

$$\mathcal{R}_{\rm M} \equiv \frac{\tau_{\rm diff}}{\tau_{\rm adv}} = \frac{\ell\,v}{\eta} \tag{96}$$

See that the regime of a high conductivity, $\sigma^{-1} \rightarrow 0$, implies $\eta \rightarrow 0$, meaning $\mathcal{R}_{\rm M} \rightarrow \infty$, or in other words $\tau_{\rm diff} \gg \tau_{\rm adv}$. That is, **in the perfect conductivity limit, diffusion of the magnetic field out of the system, is negligible with respect to the advection of the magnetic field with the fluid**.

In the case of the static system $v = 0$, the typical velocity that occurs in a magnetized fluid should be Alfvén velocity. With such, the magnetic Reynolds number is referred to as the **Lundquist number** .

Ideal MHD

Alfven's theorem

Assuming therefore perfect conductivity and non-relativistic bulk velocities, we have

$$\partial_t \vec{B} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) \tag{97}$$

which, keeping in mind the Gauss's law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$, implies the **magnetic flux conservation**, namely that the magnetic flux $\phi_{\rm M} = \int \vec{B} \cdot d\vec{\mathcal{S}}$ through a surface $\mathcal{S}$ moving with the bulk fluid velocity, is constant, $D_t \, \phi_{\rm M} = 0$ (**Alfven's theorem**, or the "frozen-in flux" theorem).

An another way of looking at it, is to use this simplified induction equation (with no diffusion term) combined with the continuity (mass conservation) equation $\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$, yielding

$$D_t \, \frac{\vec{B}}{\rho} = \left(\frac{\vec{B}}{\rho} \cdot \vec{\nabla} \right) \, \vec{v} \tag{98}$$

which implies the **field line conservation**, namely that changes of magnetic field per unit mass, B/ρ, along a fluid trajectory, are due only to the stretching or orientation of field lines caused by fluid motion.

Ideal MHD

Total energy conservation

Let us now consider the time component of the total energy-momentum conservation, i.e., the energy equation $\nabla_\mu\,(\mathcal{T}^{\mu 0} + \mathcal{T}^{\mu 0}_{\mathrm{EM}}) = 0$. In Cartesian coordinates it can be written in the **conservation form** as

$$\partial_t\;(U + U_{\mathrm{EM}}) + \vec{\nabla}\cdot\left[(U+p)\vec{v} + \vec{P}_{\mathrm{EM}}\right] = 0 \tag{99}$$

where, as defined previously, the fluid total energy density $U = w\Gamma^2 - p$, the EM field energy density $U_{\mathrm{EM}} = (E^2 + B^2)/8\pi$, and the Poynting flux $\vec{P}_{\mathrm{EM}} = c\,(\vec{E}\times\vec{B})/4\pi$.

Using the Poynting theorem $\partial_t\,U_{\mathrm{EM}} + \vec{\nabla}\cdot\vec{P}_{\mathrm{EM}} = -\vec{j}\cdot\vec{E}$, the above is equivalent to

$$\partial_t\,U + \vec{\nabla}\cdot(U+p)\vec{v} = \vec{j}\cdot\vec{E} \tag{100}$$

However, **for an ideal fluid, there can be no energy dissipation, including any conversion of EM energy into the internal energy of the fluid, so $\vec{j}\cdot\vec{E}$ must vanish!** In other words, in the ideal MHD regime, the conservation law for the total energy density reduces to that of the fluid alone.

Ideal MHD

Ohmic dissipation

Recall first that for **ideal (non-dissipative)** and non-relativistic fluid itself, the energy conservation $\partial_t \tilde{U} + \vec{\nabla} \cdot (\tilde{U} + p)\vec{v} = 0$ is equivalent to the specific entropy conservation, $D_t s = 0$.

So we see that $\vec{j} \cdot \vec{E} \neq 0$ would imply $D_t s \neq 0$...

Now, as an example, consider the non-relativistic regime, where $\vec{E} \simeq \sigma^{-1}\vec{j} - \vec{\beta} \times \vec{B}$, and therefore

$$\vec{j} \cdot \vec{E} \simeq \frac{1}{\sigma} j^2 - \vec{j} \cdot \left(\vec{\beta} \times \vec{B}\right) \xrightarrow{\sigma^{-1} \to 0} \vec{\beta} \cdot \vec{F}_{\mathrm{L}} = 0 \tag{101}$$

That is, in the regime of perfect conductivity ($\sigma^{-1} \to 0$), vanishing of the product $\vec{j} \cdot \vec{E}$ — meaning that $\vec{E} \perp \vec{j}$ — is guaranteed self-consistently. This is because, in ideal MHD, the electric field in the fluid rest frame vanishes. As a result, the EM field does no irreversible work on the fluid; the Lorentz force, which in this regime has only a magnetic component, $\vec{F}_{\mathrm{L}} = c^{-1}\vec{j} \times \vec{B}$, can alter the fluid's bulk motion or redistribute internal energy, but it does not contribute to heating or entropy production.

Only when resistivity is finite, $\sigma \neq 0$, allowing for an electric field component parallel to the current $\vec{E} \parallel \vec{j}$, can Ohmic dissipation occur, with $\vec{j} \cdot \vec{E} > 0$. That is, non-zero plasma resistivity provides a dissipative sink for magnetic field energy: **as magnetic energy diffuses out of the system, it decreases over time, and the entire loss is converted into Ohmic heating of the fluid, with $\rho T D_t s = \sigma^{-1} j^2$.**

In an analogous way, when considering space components of the total energy-momentum conservation, i.e., the momentum equation $\nabla_\mu \left(\mathcal{T}^{\mu k} + \mathcal{T}^{\mu k}_{\mathrm{EM}}\right) = 0$, we obtain the **conservation form**

$$\partial_t \left(P^k + \frac{1}{c^2} P^k_{\mathrm{EM}} \right) + \partial_i \left(P^k v^i + p\delta^{ik} + \Pi^{ki}_{\mathrm{EM}} \right) = 0 \tag{102}$$

where the Maxwell stress tensor $\Pi^{ik}_{\mathrm{EM}} = -\frac{1}{4\pi}\left(E^i E^k + B^i B^k\right) + \frac{1}{8\pi}\left(E^2 + B^2\right)\delta^{ik}$. Using the momentum equation for EM field, the above is equivalent to

$$\partial_t P^k + \partial_i \left(P^k v^i + p\delta^{ik} \right) = F^k_{\mathrm{L}} \tag{103}$$

From here it follows that the EM field is acting dynamically on the fluid through the Lorentz force $\vec{F}_{\mathrm{L}}$.
This leads to changes in the fluid's bulk kinetic energy and/or pressure, but does not increase its entropy.

Neglecting the displacement current ($c^{-1}\partial_t\vec{E} = 0$) and assuming electric neutrality of a fluid ($Q = 0$) — both of which are justified in the non-relativistic regime, as elaborated above — the Lorentz force becomes

$$\begin{aligned} \vec{F}_{\mathrm{L}} = \frac{1}{c}\vec{j} \times \vec{B} &= \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} = \\ &= \frac{1}{4\pi}(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right) . \end{aligned} \tag{104}$$

The first term on the right-hand side represents the magnetic tension force, which acts along curved field lines ("tension pulls"), while the second term corresponds to the magnetic pressure gradient, which acts perpendicular to field lines where magnetic pressure varies ("pressure pushes").

Ideal non-relativistic MHD: main assumptions

Let's clarify the main assumptions behind "non-relativistic ideal MHD approximation":

- **non-relativistic bulk velocities** $\beta \ll 1$

  $\rightarrow \quad \partial_t \vec{E} \ll 4\pi \vec{j} \quad$ and $\quad Q\vec{E} \ll c^{-1}\, \vec{j} \times \vec{B}$

  $\rightarrow \quad \vec{j} \simeq (c/4\pi)\,(\vec{\nabla} \times \vec{B}) \quad$ and $\quad \vec{F}_{\rm L} \simeq c^{-1}\, \vec{j} \times \vec{B}$

- **perfect conductivity regime** $\sigma^{-1} \rightarrow 0$

  $\rightarrow \quad \vec{E}' = 0$

  $\rightarrow \quad \vec{E} \simeq -\vec{\beta} \times \vec{B} \quad$ and $\quad \vec{j} \cdot \vec{E} = 0$

- **both** $\beta \ll 1$ **and** $\sigma^{-1} \rightarrow 0$

  $\rightarrow \quad \mathcal{R}_{\rm M} \gg 1$

  $\rightarrow \quad \partial_t \vec{B} \simeq \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)$

All in all, the set of **non-relativistic ideal (perfect conductivity) MHD equations for a polytropic fluid** reads as

$$
\begin{aligned}
\partial_t \rho &= -\vec{\nabla}(\rho \vec{v}) && (105)\\
D_t\left(\frac{p}{\rho^{\hat{\gamma}}}\right) &= 0 && (106)\\
\rho\, D_t \vec{v} &= -\vec{\nabla} p + \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} && (107)\\
\partial_t \vec{B} &= \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) && (108)
\end{aligned}
$$

with the boundary/initial condition $\vec{\nabla} \cdot \vec{B} = 0$.

In the ideal non-relativistic MHD, electric field becomes a secondary quantity!

The above system of ideal non-relativistic MHD equations, can be cast in the **conservation form**, where we typically ignore the electric terms in the Maxwell stress tensor $\hat{\Pi}_{\mathrm{EM}}$ and in the field energy density U_{EM}, as well as the time variation of the Poynting flux $\partial_t \vec{P}_{\mathrm{EM}}$. As a result, we obtain:

$$\partial_t \rho + \partial_i \left[\rho v^i\right] = 0 \tag{109}$$

$$\partial_t \left(\frac{1}{2}\rho v^2 + \varepsilon + \frac{1}{8\pi}B^2\right) + \partial_i \left[\left(\frac{1}{2}\rho v^2 + \varepsilon + p + \frac{1}{8\pi}B^2\right) v^i - \frac{1}{4\pi} v_j B^i B^j\right] = 0 \tag{110}$$

$$\partial_t \left(\rho v^k\right) + \partial_i \left[\rho v^i v^k + \left(p + \frac{1}{8\pi}B^2\right)\delta^{ik} - \frac{1}{4\pi}B^i B^k\right] = 0 \tag{111}$$

$$\partial_t B^k + \partial_i \left[v^i B^k - v^k B^i\right] = 0 \tag{112}$$