Lecture 2:
Particle Acceleration

i) General problem
ii) Rotating magnetospheres, reconnection
iii) Fermi process
iv) Turbulent acceleration: momentum diffusion
v) Shock (diffusive) acceleration: momentum advection

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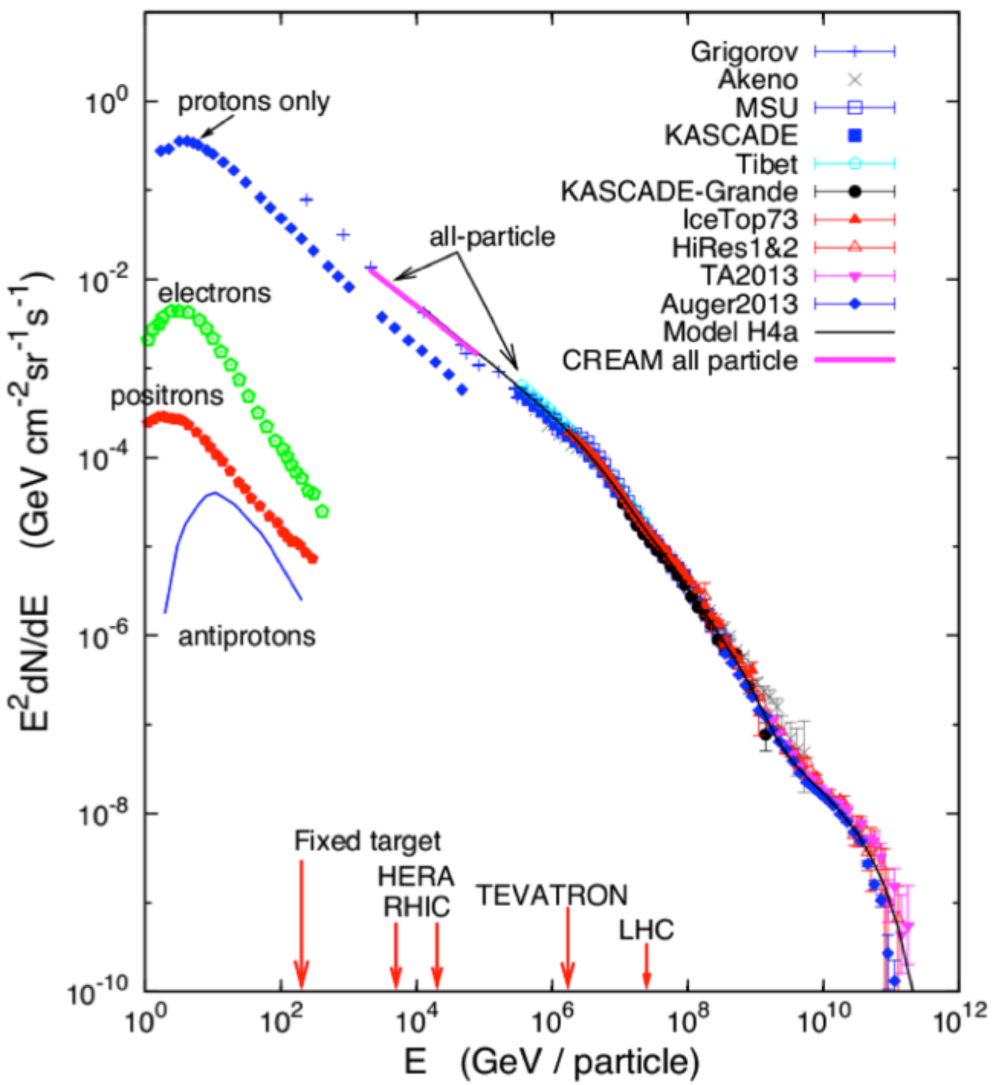
What We Need to Explain



What we need to explain:

a tiny fraction of particles is extracted from a "thermal pool", and accelerated to extremely high, ultra-relativistic energies, forming a universal power-law spectrum

$N(E) \equiv dN(E)/dE \propto E^{-s}$ with $s \simeq 2$

this spectrum is subsequently modified due to propagation effects, to $N(E) \propto E^{-\tilde{s}}$ with $s < \tilde{s} < 3$ depending on local plasma conditions (radiative cooling, etc.).

Astrophysical Plasma

Astrophysical plasma is typically low-density, highly ionised, and almost always magnetised.

Collisionless plasma means that, due to extremely low plasma densities, mean free path for Coulomb (binary) scattering is much larger than scales of the systems**:** $\lambda_C \gg \ell$

However, **continuous character** of the plasma is assured by the magnetic field, without action of the Coulomb scatterings — magnetic field is always strong enough to assure that the effective mean free path of particles, in the limiting case the particle gyroradius, is always much smaller than the scale of the system: $\ell \gg r_g$

Moreover, collisionless and ionised plasma is **highly conductive**, meaning that the Debye screening length is much smaller than the sale of the plasma: $\ell \gg \lambda_D$

General Problem

Since collisonless, ionised and magnetised and plasma is highly conductive, there is no large-scale stable E field, only B field; and B fields alone cannot accelerate particles…

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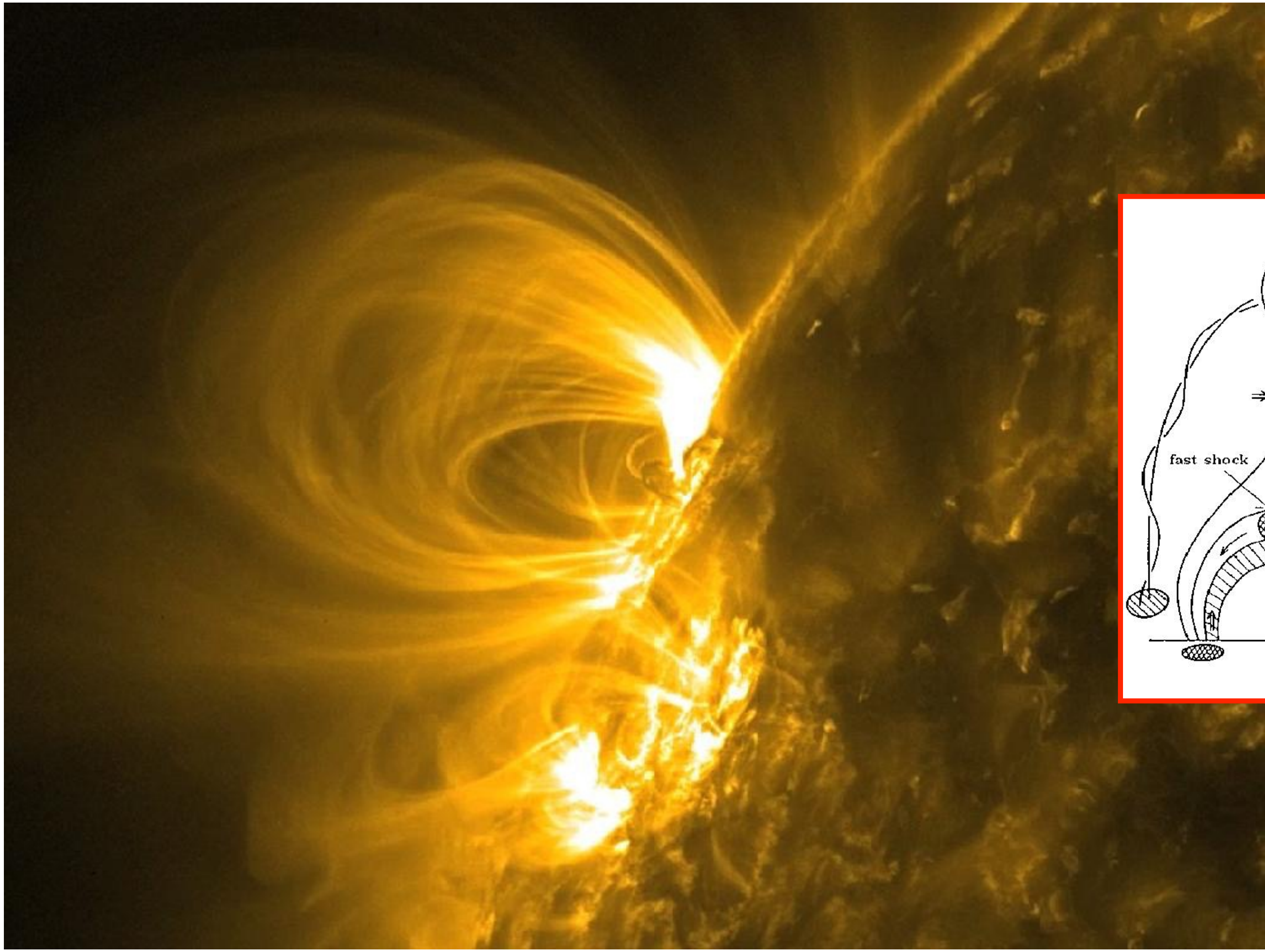
→ Acceleration in time-varying magnetic field (rotating magnetospheres; magnetic reconnection regions) *[Faraday]*

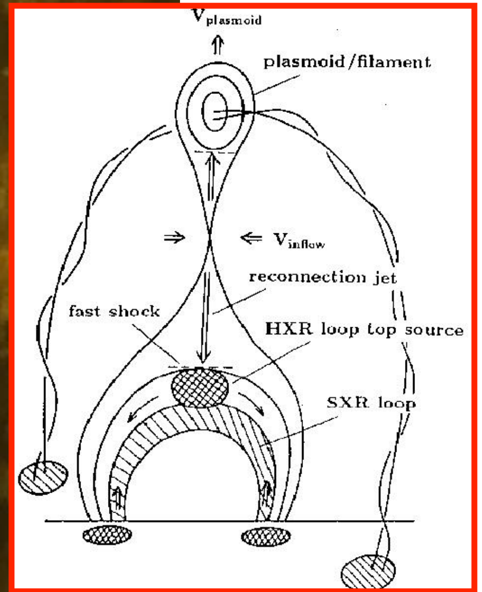
$$\frac{1}{c}\partial_t \overrightarrow{B} = -\overrightarrow{\nabla} \times \overrightarrow{E}$$

→ Relative velocity of magnetised bodies (Fermi acceleration processes) *[ideal Ohm]*

$$\overrightarrow{E} = -\frac{1}{c}\vec{v} \times \overrightarrow{B}$$

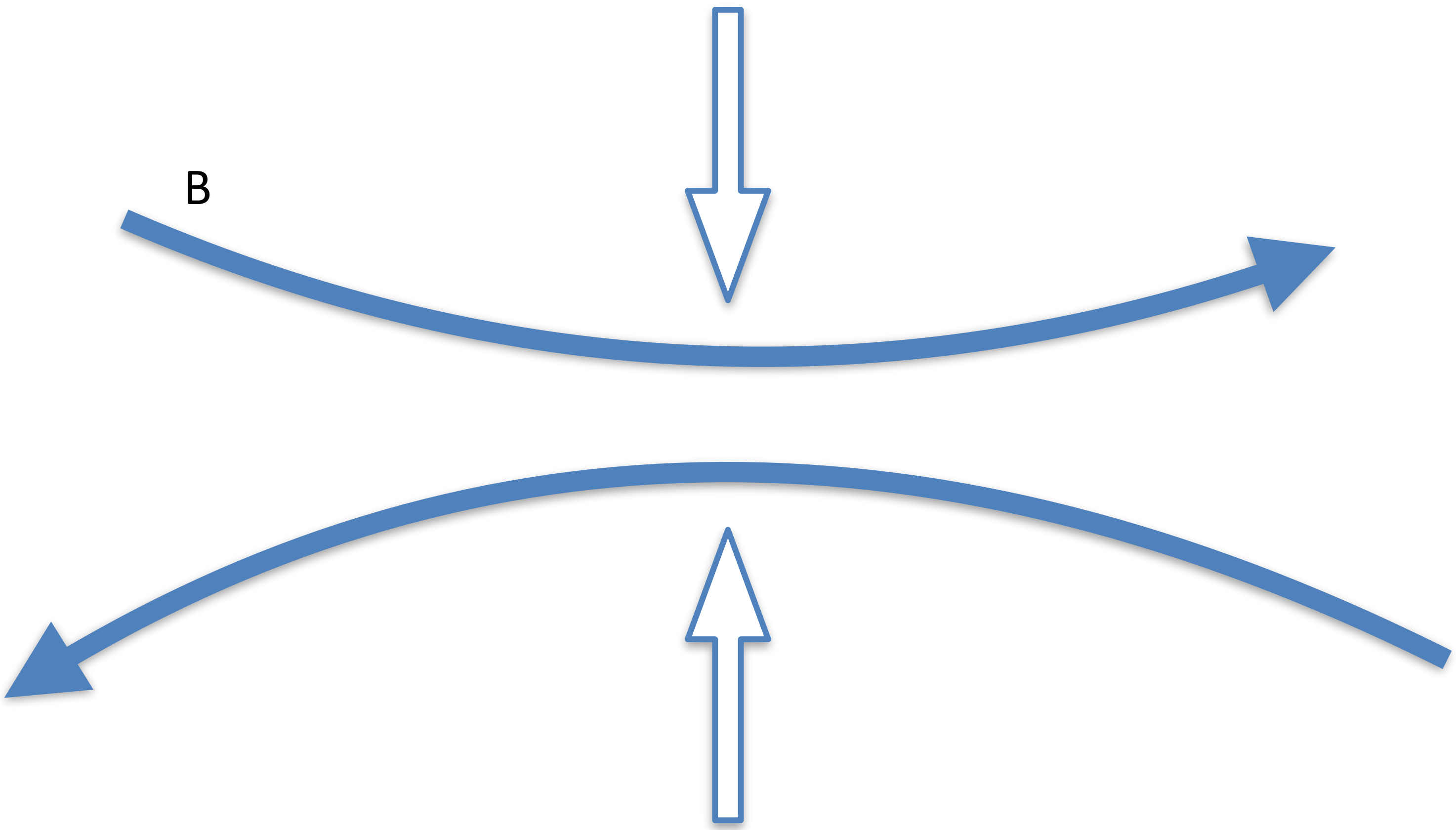
Magnetic Reconnection





P.A. Sweet (1958), E.N. Parker (1957-63)

Reconnection: General Idea



Reconnection: General Idea

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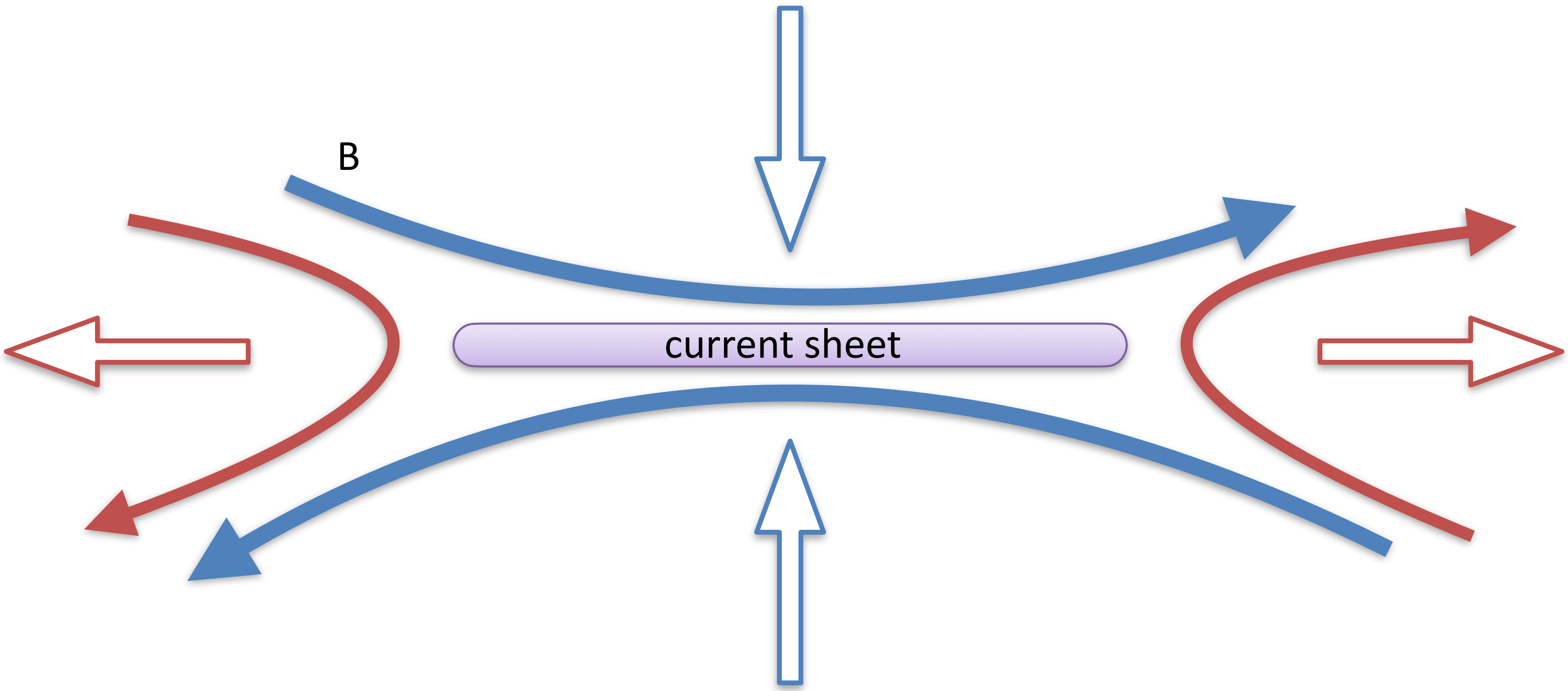
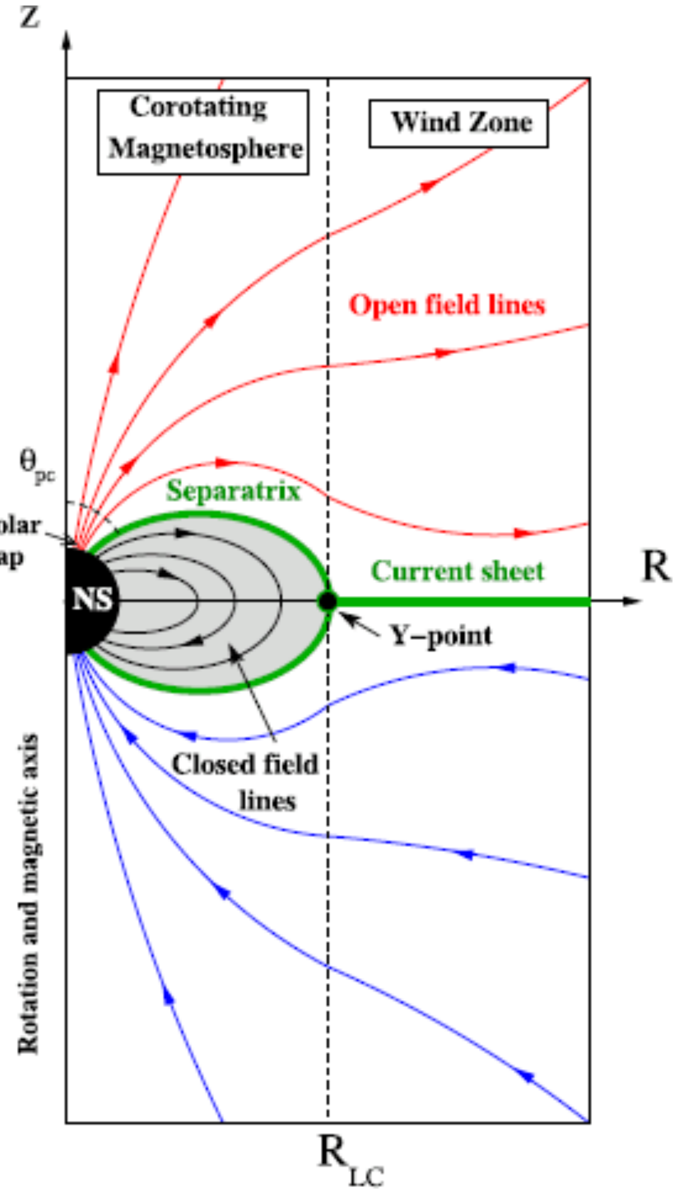
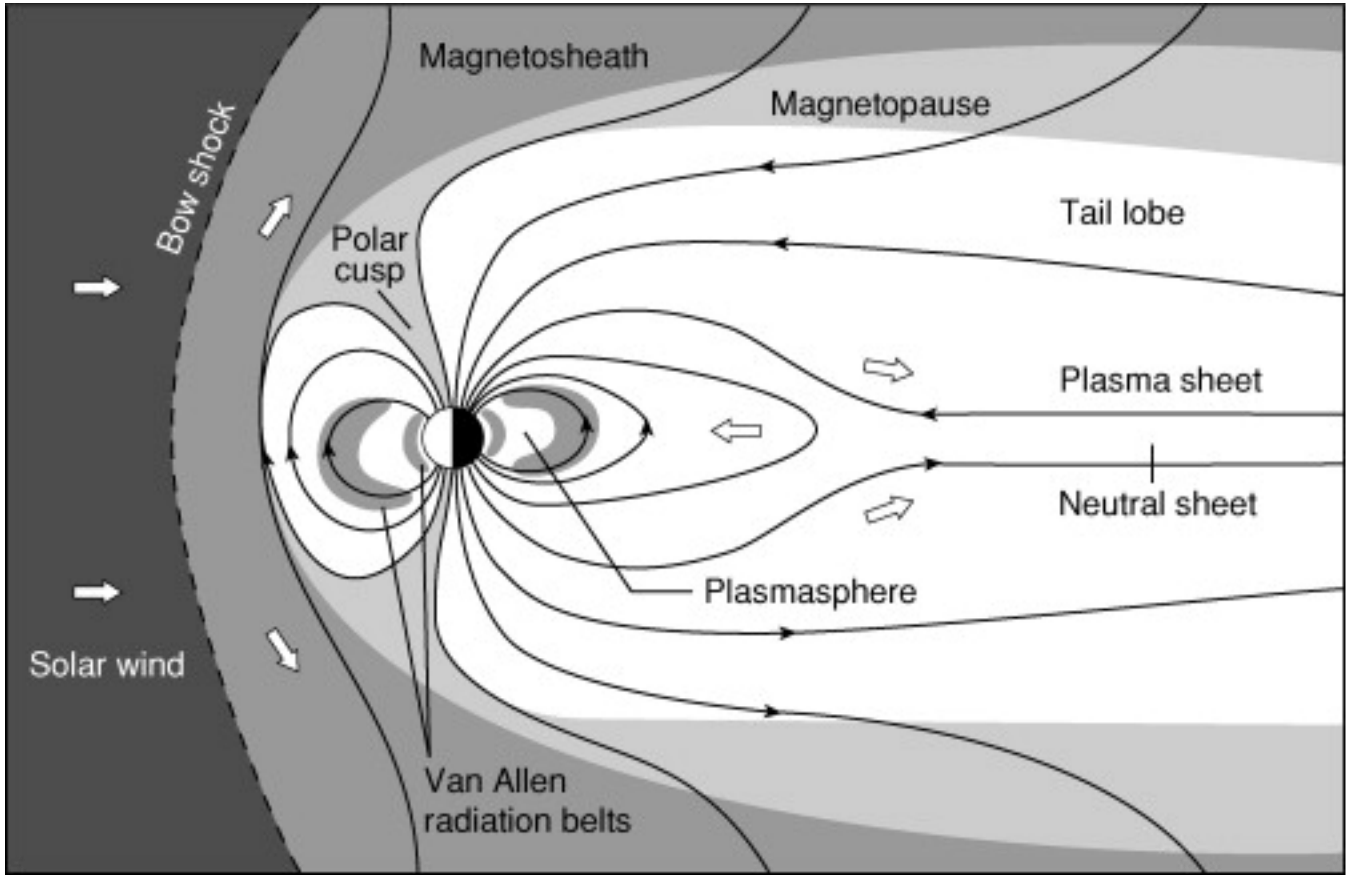


Rotating Magnetospheres

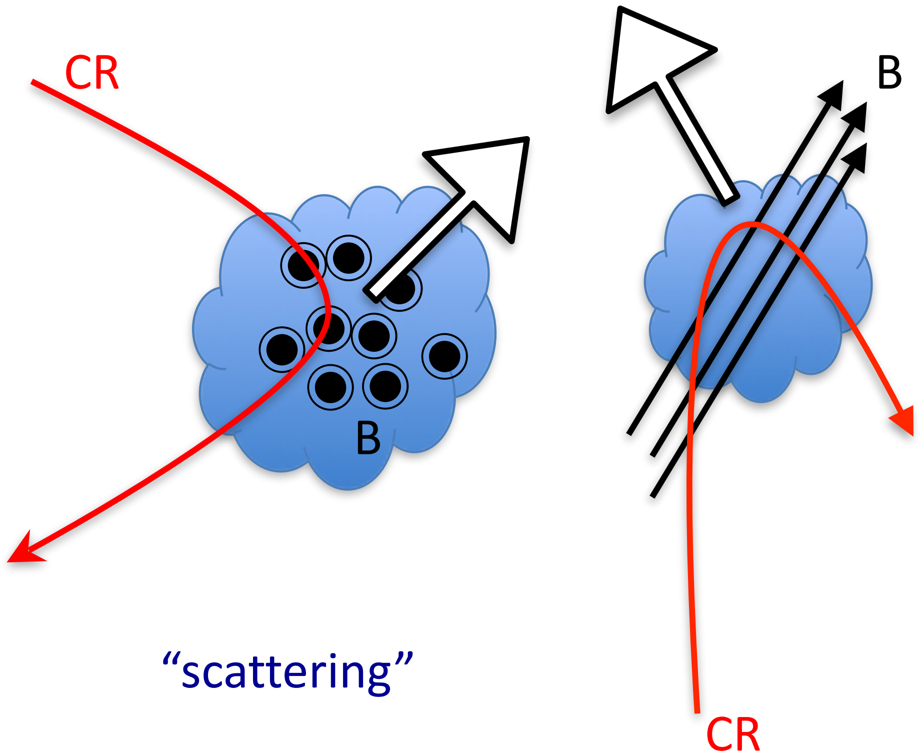




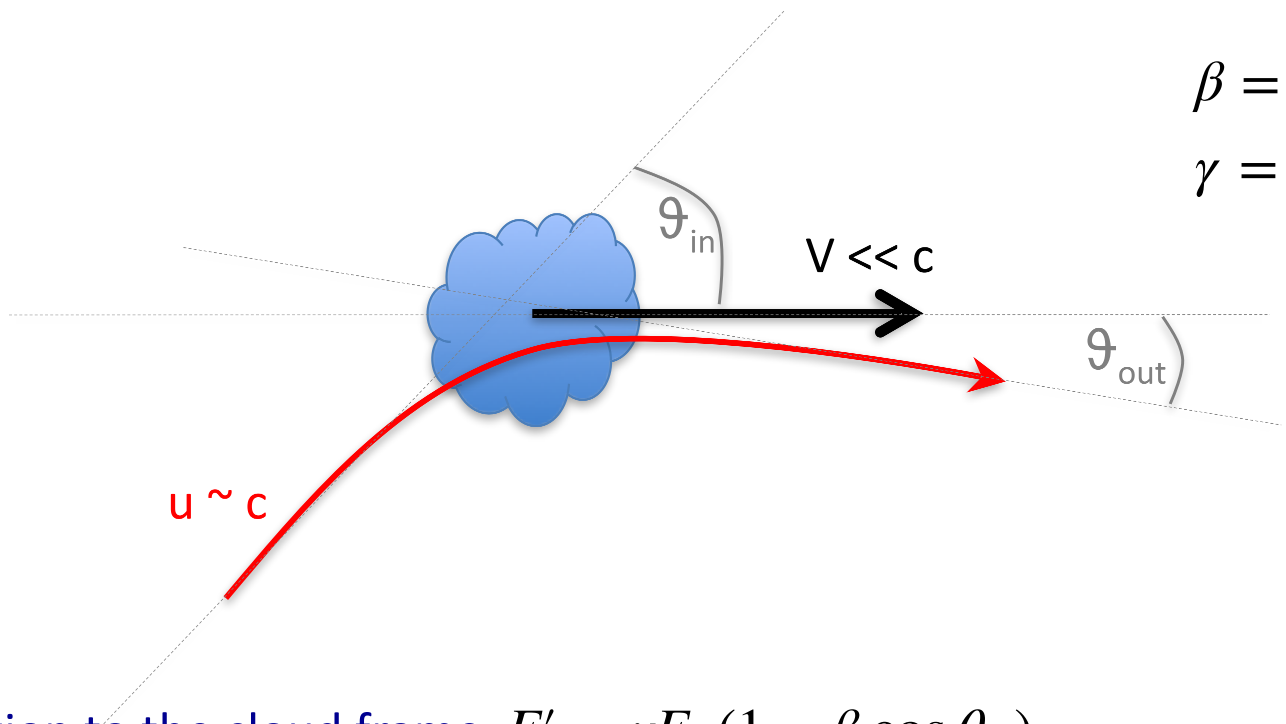
Fermi Process

Fermi proposal (1949):

Galaxy is filled with randomly moving clouds of gas with frozen-in magnetic field; high-energy charged particles can "scatter off" these magnetized clouds.



Fermi Process



$\beta = V/c$

$\gamma = (1 - \beta^2)^{-1/2}$

Transformation to the cloud frame $E_0' = \gamma E_0 (1 - \beta \cos \theta_{in})$

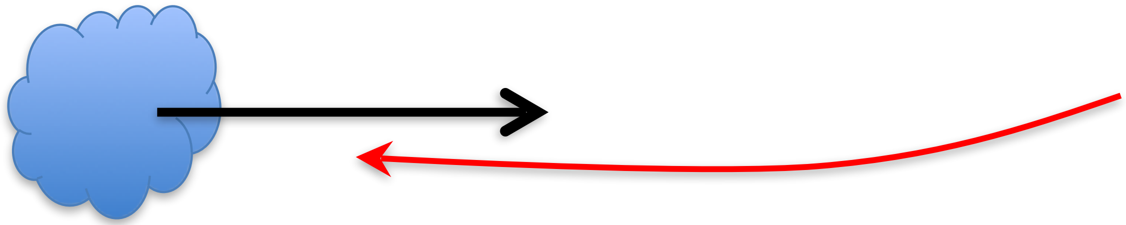
Elastic scattering in the cloud rest frame $E' = E_0'$

Transformation to the observer frame $E = \gamma E' (1 + \beta \cos \theta'_{out})$

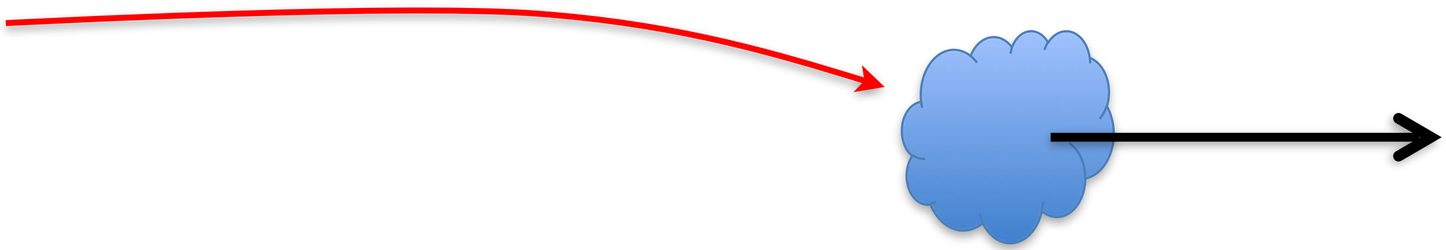
Energy gains or losses depending on ϑ_{in}

Fermi Process

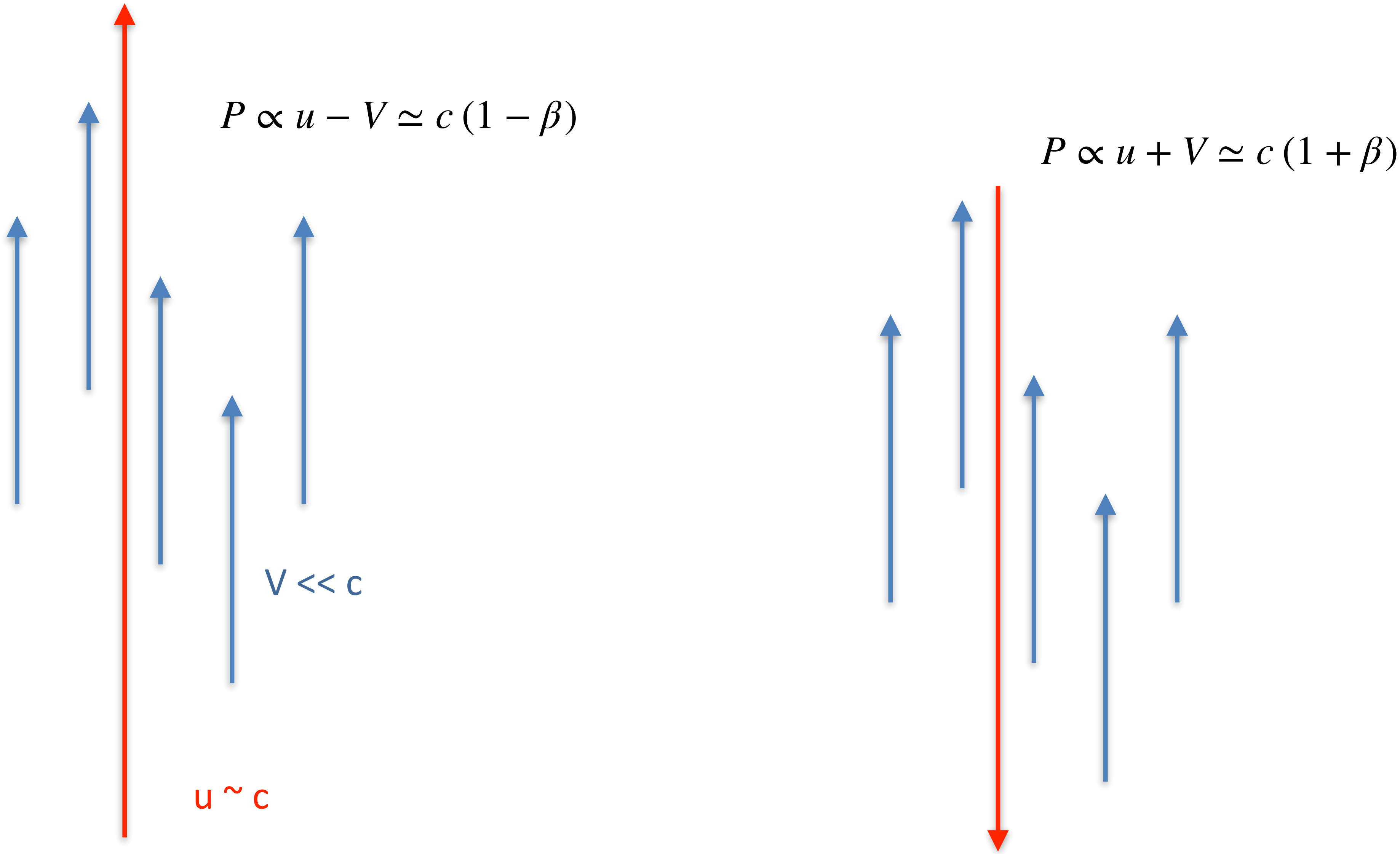
"Head-on collision"
Energy GAIN: $E>E_0$



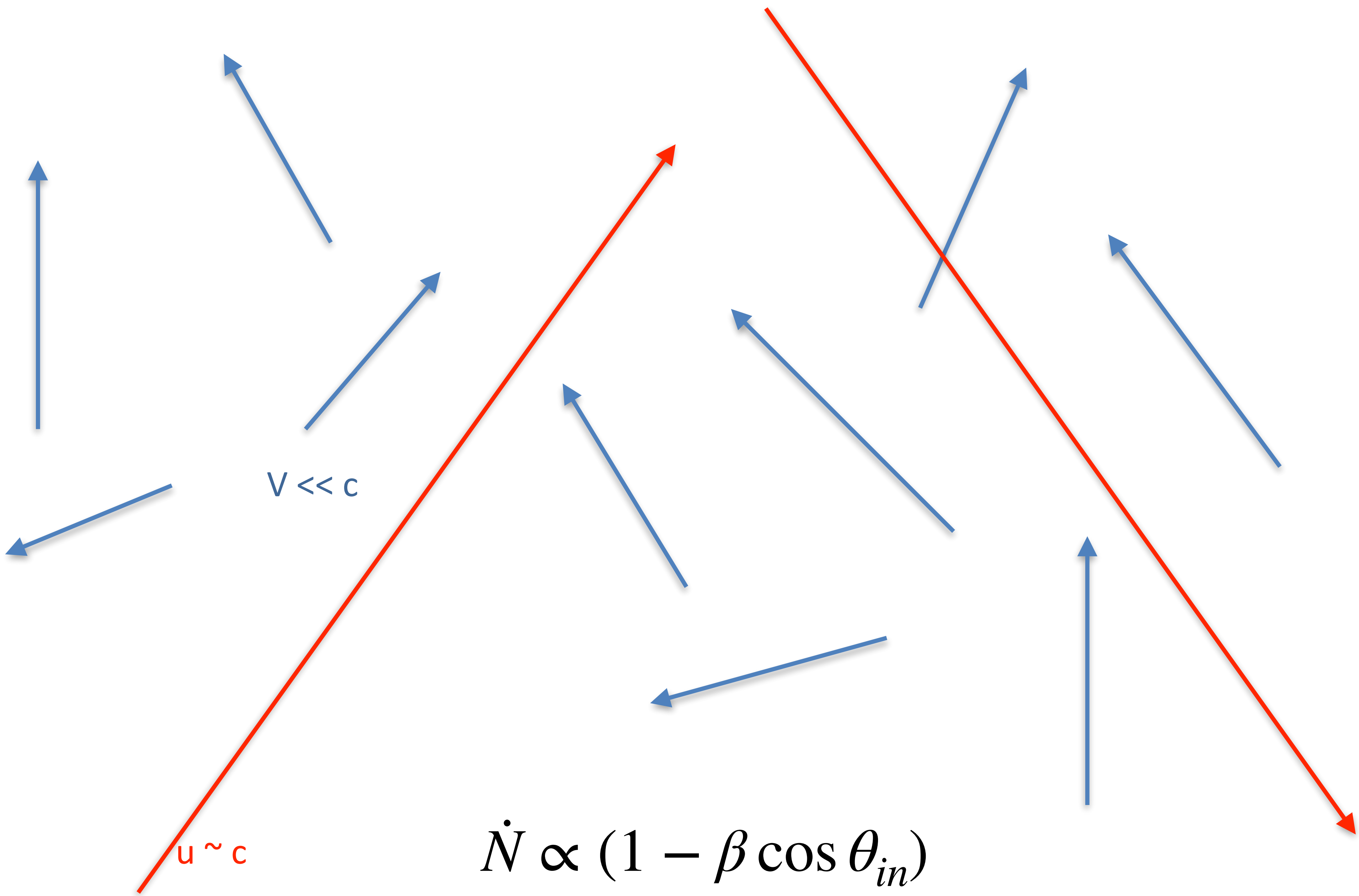
"Tail-on collision"
Energy LOSS: $E<E_0$



Fermi Process



Fermi Process



$$\dot{N} \propto (1 - \beta \cos \theta_{in})$$

Fermi Process

$\mu \equiv \cos\theta_{in}$
$\tilde{\mu}' \equiv \cos\theta'_{out}$
$\dot{N} = 1 - \beta\mu$

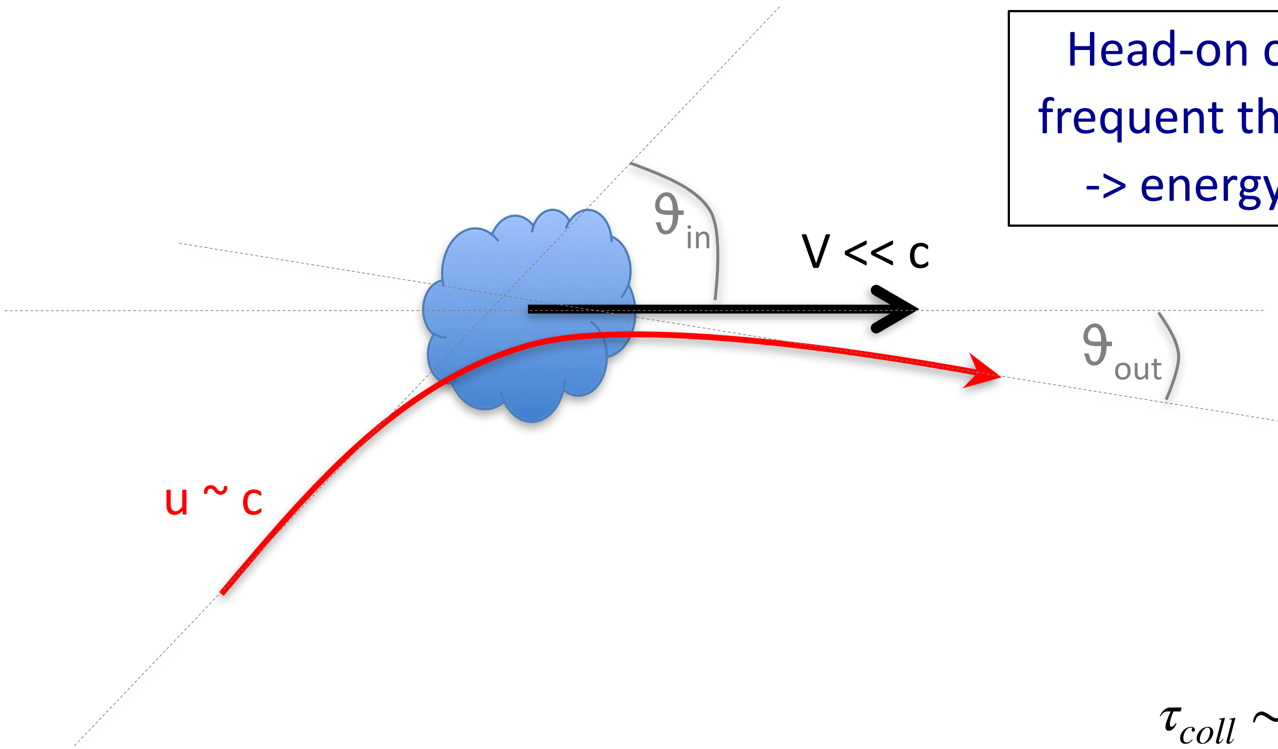
$$\frac{\Delta E}{E} \equiv \frac{E - E_0}{E_0} = \gamma^2 (1 - \beta\mu)(1 + \beta\tilde{\mu}') - 1 = \gamma^2(1 - \beta\mu) + \beta\gamma^2(1 - \beta\mu)\,\tilde{\mu}' - 1$$

angle averaging
$$\langle\frac{\Delta E}{E}\rangle = \frac{\int_{-1}^{+1} d\mu\ (\Delta E/E)\ \dot{N}}{\int_{-1}^{+1} d\mu\ \dot{N}}$$

we assume that μ and $\tilde{\mu}'$ are uncorrelated, so that $\langle(1 - \beta\mu)\,\beta\tilde{\mu}'\rangle = \langle(1 - \beta\mu)\rangle \cdot \langle\beta\tilde{\mu}'\rangle$, and also that $\langle\tilde{\mu}'\rangle = 0$, corresponding to the situation when scattered particle "forgets" about its initial direction.

$$\langle\frac{\Delta E}{E}\rangle = \frac{4}{3}\gamma^2\beta^2 \quad \rightarrow \quad \sim \beta^2$$

Fermi Process



Head-on collisions are more frequent than tail-on collisions -> energy gain ***on average***

$$\tau_{coll} \sim \frac{\lambda}{c}$$

$$\frac{dE}{dt} \simeq \langle \frac{\Delta E}{E} \rangle \frac{E}{\tau_{coll}} \sim \frac{\beta^2 E c}{\lambda} \equiv \frac{E}{\tau_{acc}}$$

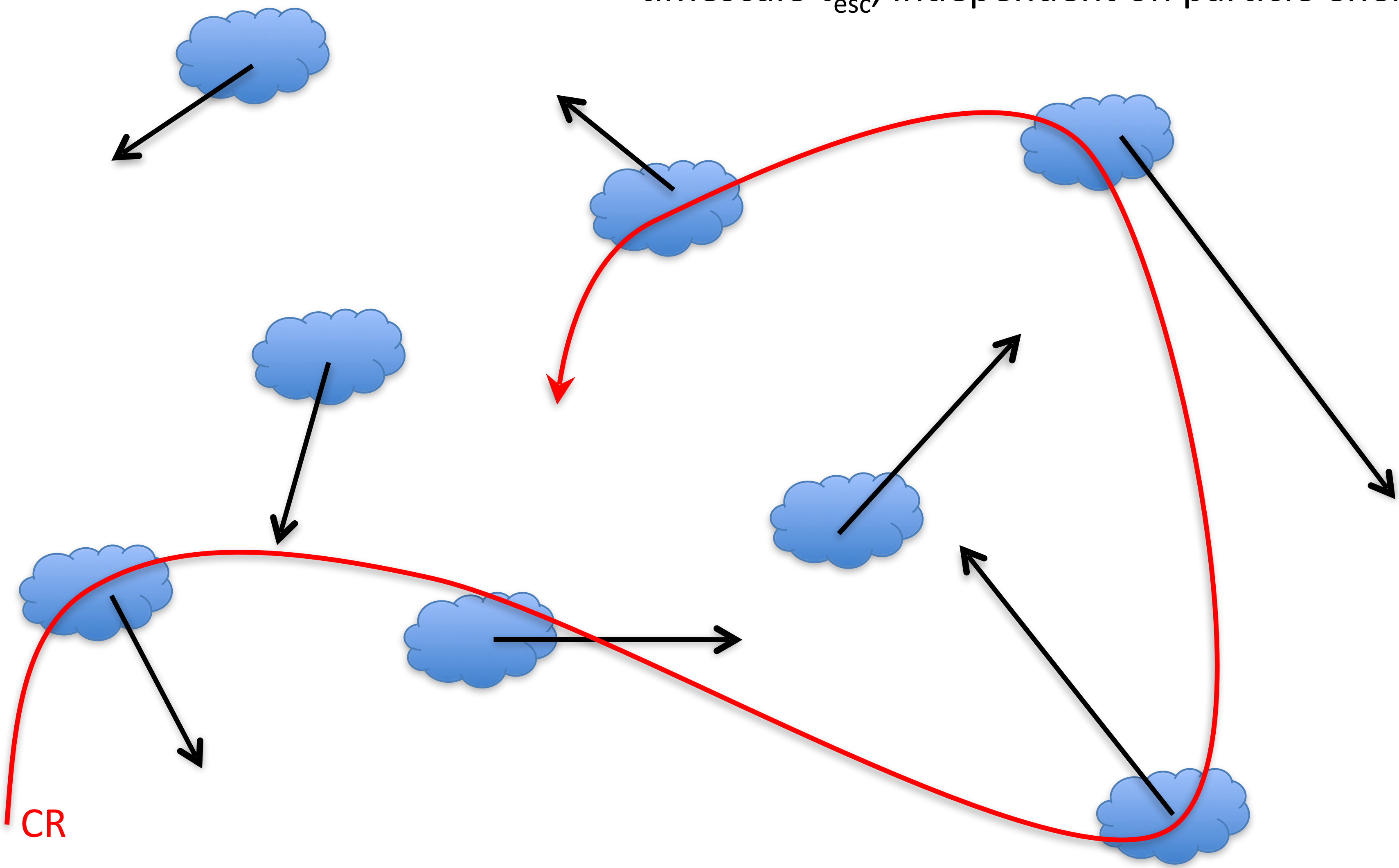
$$\rightarrow \quad \tau_{acc} \sim \frac{\lambda}{c} \beta^{-2}$$

2nd order in β, so slow in the case of non-relativistic velocities of scattering centers;

Process ***may*** results in a power-law distribution of the accelerated particles, but with energy index depending on the τ_{acc}/τ_{esc} ratio.

Fermi Process

Fermi assumed scattering mean free path λ, and so the acceleration timescale τ_{acc}, as well as the escape timescale τ_{esc}, independent on particle energy.



Fermi Process: Spectral Evolution

$$\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left(\frac{E}{\tau_{\rm acc}}\, n(E,t)\right) - \frac{n(E,t)}{\tau_{\rm esc}}$$

describes “regular” energy changes (acceleration)

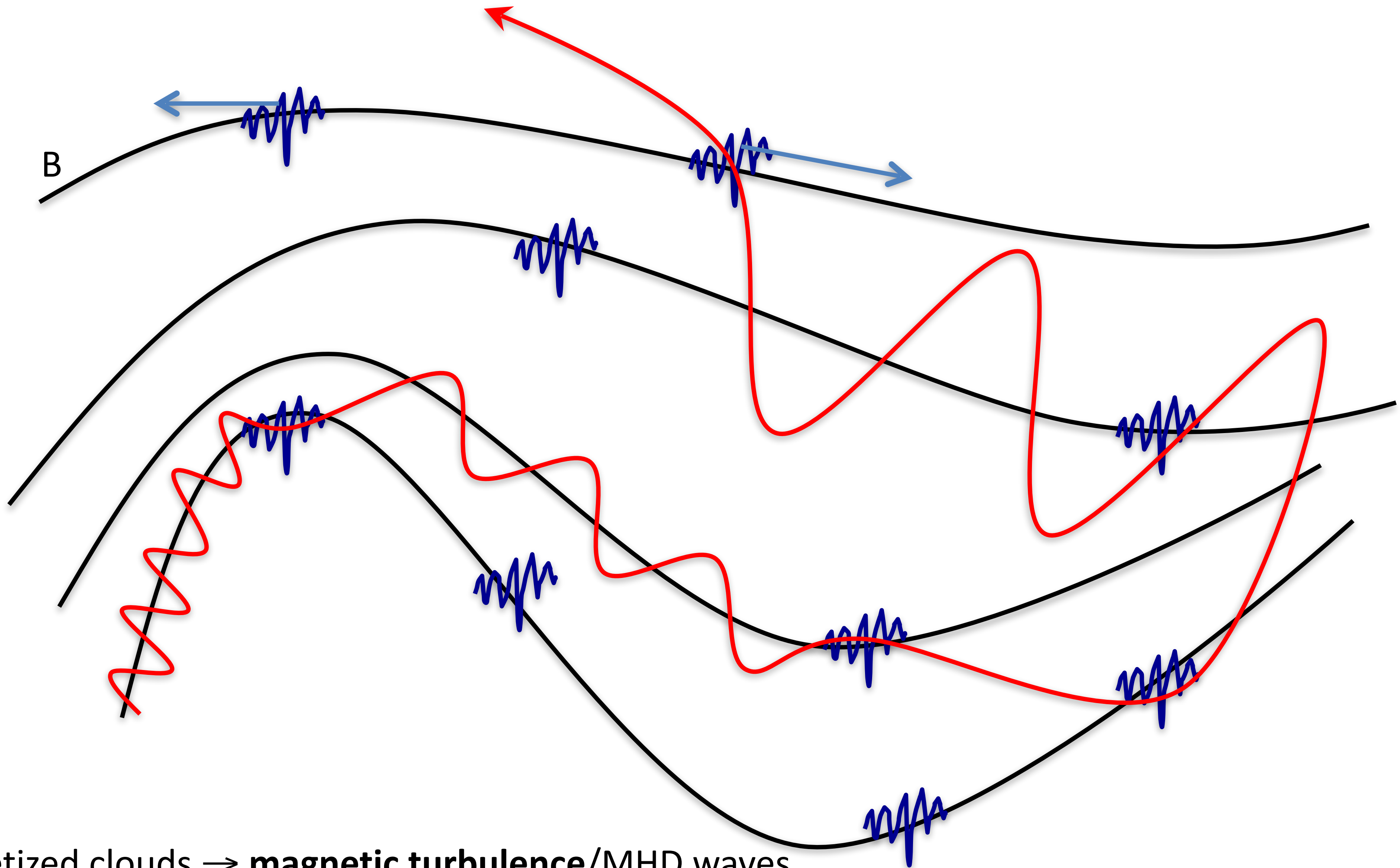
approximates particles’ escape from the system

we look for steady-state solutions $\partial_t = 0$:

$$\frac{dn(E)}{dE} = -\left(1 + \frac{\tau_{\rm acc}}{\tau_{\rm esc}}\right)\frac{n(E)}{E} \quad \Longrightarrow \quad n(E) \propto E^{-\left(1+\frac{\tau_{\rm acc}}{\tau_{\rm esc}}\right)}$$

- power-law energy distribution of accelerated particles!
- however, energy index not universal, but instead depending on the τ_{acc}/τ_{esc} ratio…
- only in the $1/\tau_{esc}$ -> 0 the energy spectrum approaches asymptotically $n(E) \sim E^{-1}$

Turbulent Acceleration

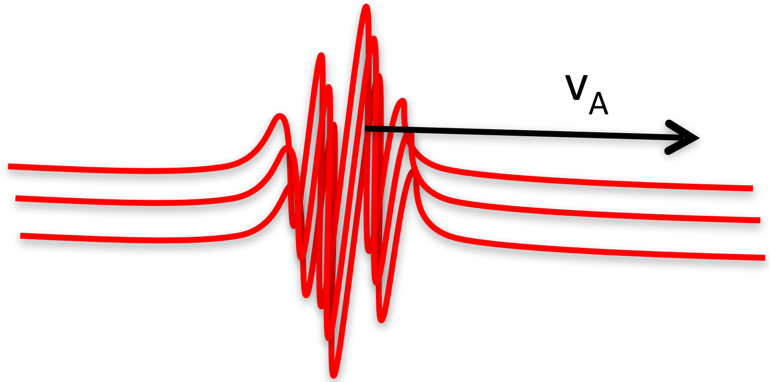


magnetized clouds → **magnetic turbulence**/MHD waves

regular acceleration → **diffusion in momentum space**

$\tau_{acc}=\tau_{acc}(E)$ and $\tau_{esc}=\tau_{esc}(E)$ related to particle-waves interactions

Particle-Wave Interactions



The key idea is that charged particles in a plasma can only effectively interact with Alfvén waves if they are in resonance. The resonance condition for wave–particle interactions is:

$$\omega - k_{\|} v_{\|} = n\Omega$$

$\omega = k_{\|} v_A$ wavefrequency

$k_{\|} = 2\pi / \lambda_{\|}$ component of wavevector along the magnetic field

$v_{\|}$ particle velocity along the magnetic field

n integer (harmonic number)

$\Omega = v_{\perp} / r_g$ particle gyrofrequency for the particle's gyroradius r_g

Particle-Wave Interactions

For Alfvén waves, the most relevant resonance is the **gyroresonance** $n = -1$, meaning that the particle is interacting with the wave at a frequency close to its gyrofrequency, adjusted for Doppler shift from its motion along the field. Gyroresonance occurs when the particle's gyroradius is comparable to the Alfvén wave's parallel wavelength,

$$r_g \sim \frac{\lambda_{\|}}{2\pi}$$

If the wave is much longer than the gyroradius, the particle experiences only a slowly varying field and doesn't interact strongly. If it's much shorter, the wave averages out over a gyro-orbit.

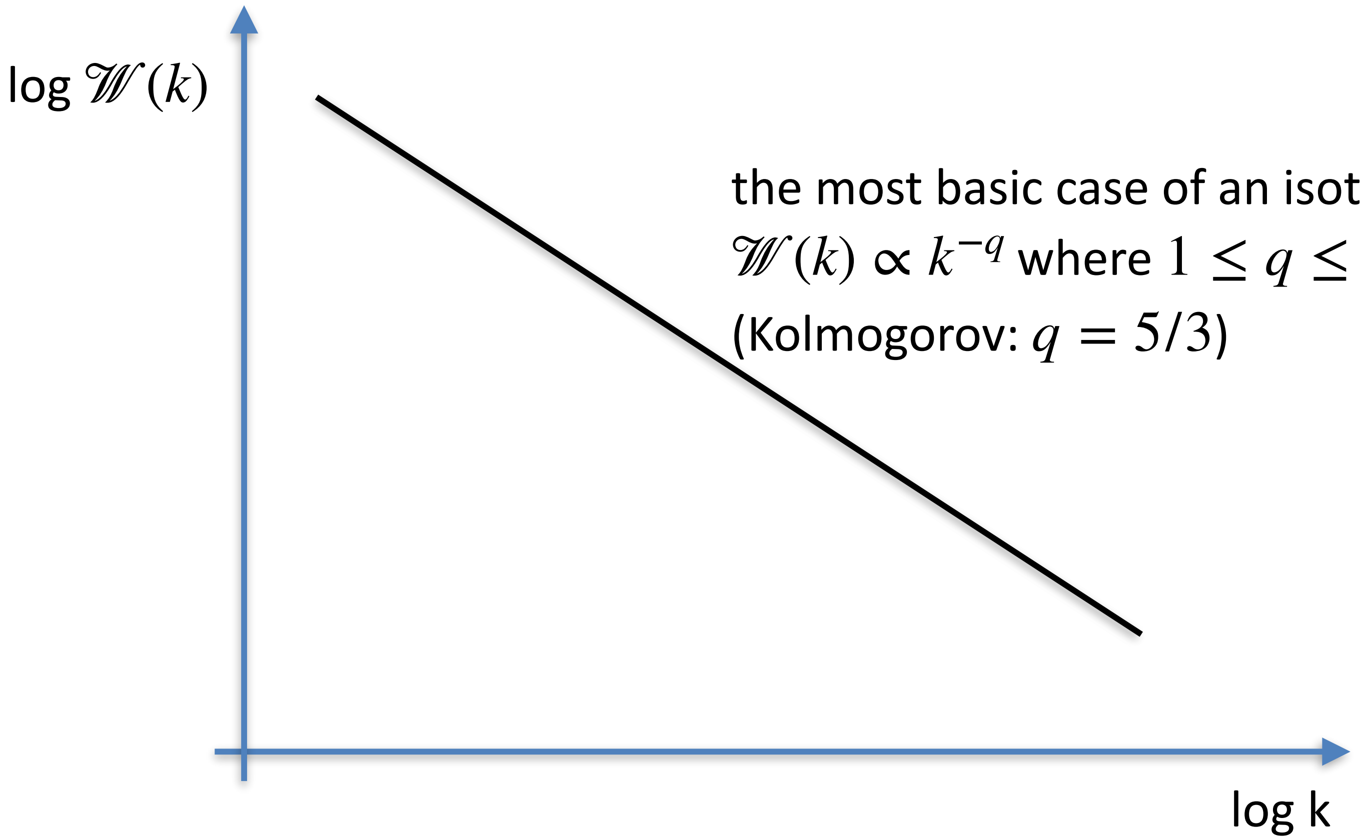
Magnetic Turbulence

Turbulence is not a simple superposition of MHD waves with various amplitudes and frequencies, but we often think in these terms…

Magnetic Turbulence

total energy in the unperturbed magnetic field $U_{B_0} = \dfrac{B_0^2}{8\pi}$

total energy in the perturbed magnetic field $U_{\delta B} = \displaystyle\int d^3\vec{k}\, \mathscr{W}(\vec{k}) = \dfrac{\delta B^2}{8\pi}$



the most basic case of an isotropic turbulence

$\mathscr{W}(k) \propto k^{-q}$ where $1 \le q \le 2$

(Kolmogorov: $q = 5/3$)

Mean Free Path

mean free path for interaction of particles with Alfvenic modes

$$\lambda(p) = r_g \left(\frac{B_0}{\delta B}\right)^2 \left(\frac{\lambda_{max}}{r_g}\right)^{q-1}$$

where the particle's gyroradius $r_g = E/eB_0$

and λ_{max} denotes maximum wavelength of Alfvenic modes for the turbulence index q

$\rightarrow$ the shortest mean free path for particle-wave interactions: $\lambda(p) \sim r_g$

corresponding to "strong turbulence" conditions $B_0 \sim \delta B$ and $q = 1$

Bohm & Hillas

in the Bohm regime one has

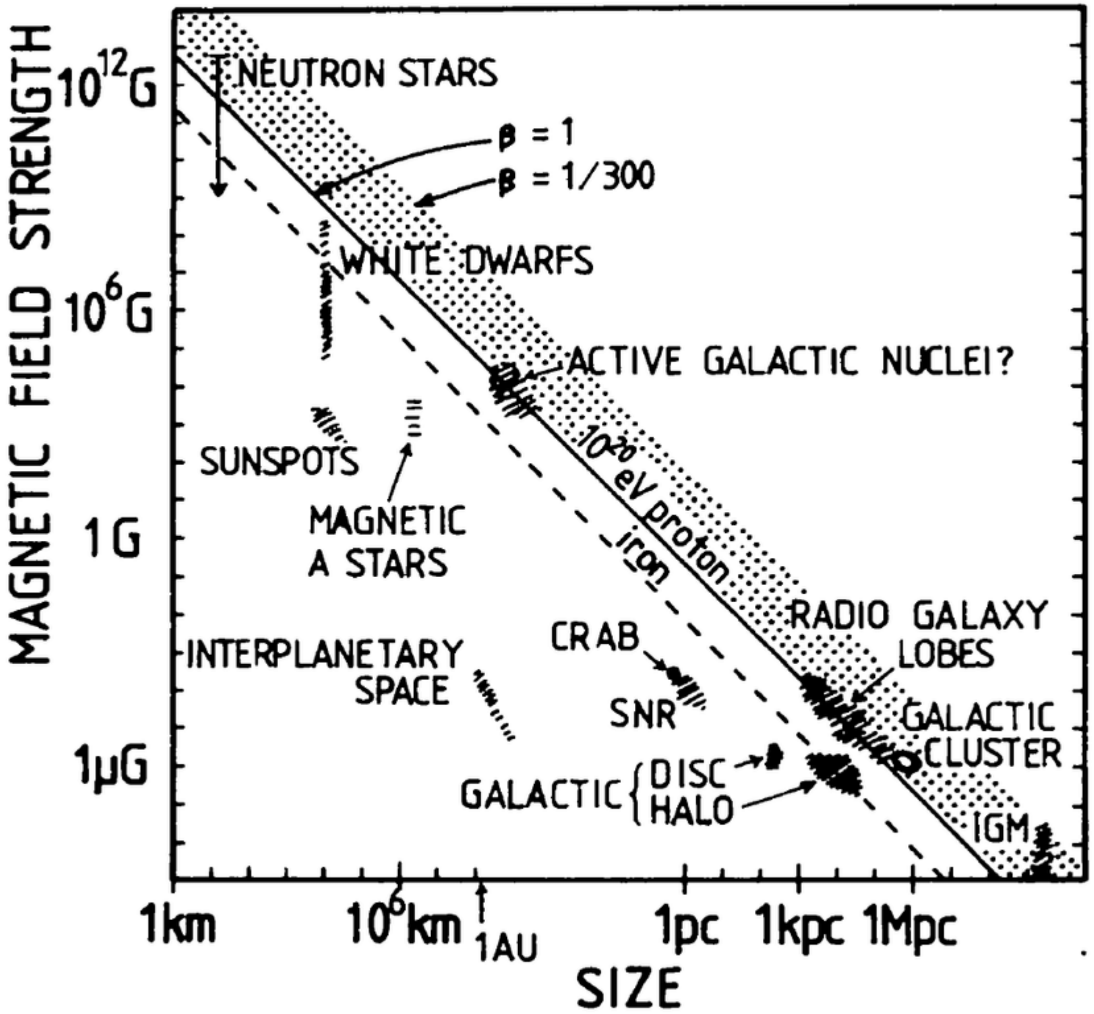
$$\lambda \sim r_g$$

and this gives us the most crude estimate of the size ***L*** and magnetisation ***B*** needed for a given system to be able to accelerate particles to the UHECR range, ***E*** ~ 1e20 eV, namely $r_g \sim L$, or

$$E \sim ZeBL$$

where ***Z*** is the atomic number of a CR

(Hillas 1984)



Vlasov Equation

Phase space density of ultrarelativistic particles: f(x, p; t)
Total number of particles: N = ∫d³x ∫d³p f(x, p; t)

For collisionless plasma, the function f(x, p; t) satisfies the relativistic Vlasov equation (colisionless Boltzmann equation) with the acceleration determined by the Lorentz force due to the electromagnetic field.

$$\frac{df(\vec{x}, \vec{p}; t)}{dt} = 0 \quad \longrightarrow$$

$$\frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt}\frac{\partial f}{\partial \vec{x}} + \frac{d\vec{p}}{dt}\frac{\partial f}{\partial \vec{p}} = 0 \quad \longrightarrow$$

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{p}} = 0}$$

Diffusion Approximation

Approx. 1: Presence of only a small-amplitude turbulence (δE, δB) in addition to the large-scale magnetic field $B_0 \gg \delta B$, such that the total plasma fields are $B = B_0 + \delta B$, and $E = \delta E$ (due to high conductivity we expect $E_0 = 0$)

Approx. 2 ("test-particle approach"): Configuration of the electromagnetic field is given, and is independent on the considered particles (their energy spectra).

Ensemble-averaging, $f(x, p; t) = \langle f \rangle + \delta f$, $\langle \delta B \rangle = \langle \delta E \rangle = 0$, and linearization of the Vlasov equation → Fokker-Planck equation for $\langle f \rangle$

Approx. 3 (diffusion approximation): $\langle f \rangle$ is spatially uniform and isotropic in p, background plasma is uniform, and velocities of turbulent modes δB are non-relativistic

Fokker-Planck equation → momentum diffusion equation

Momentum Diffusion

$$\frac{\partial \langle f \rangle}{\partial t} + \vec{U} \cdot \vec{\nabla} \langle f \rangle = \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_p \frac{\partial \langle f \rangle}{\partial p} \right) + \vec{\nabla} \cdot \left(\kappa \, \vec{\nabla} \langle f \rangle \right)$$

Convection in position space **momentum diffusion** **spatial diffusion**

stochastic acceleration = turbulent acceleration = 2^{nd} Fermi acceleration = MOMENTUM DIFFUSION

$$D_p = \frac{1}{3} \left(\frac{v_A}{c} \right)^2 \left(\frac{c}{\lambda} \right) p^2$$

$$\kappa = \frac{1}{3} c \lambda$$

$$\tau_{acc} \equiv \frac{p^2}{D_p} = 3 \frac{\lambda}{c} \beta_A^{-2}$$

$$\tau_{esc} \equiv \frac{L^2}{\kappa}$$

Momentum Diffusion

$$\frac{\partial \langle f \rangle}{\partial t} + \vec{U} \cdot \vec{\nabla} \langle f \rangle = \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_p \frac{\partial \langle f \rangle}{\partial p} \right) + \vec{\nabla} \cdot \left(\kappa \, \vec{\nabla} \langle f \rangle \right)$$

$$n(p,t) \equiv 4\pi p^2 \langle f \rangle \quad , \quad \vec{U} = 0$$

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial p} \left(D_p \frac{\partial n}{\partial p} \right) - \frac{\partial}{\partial p} \left(\frac{2 D_p}{p} n \right) + \vec{\nabla} \cdot \left(\kappa \, \vec{\nabla} n \right)$$

Momentum Diffusion

second-order (stochastic) term

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial p}\left(D_p \frac{\partial n}{\partial p}\right) - \frac{\partial}{\partial p}\left(\frac{2D_p}{p} n\right) + \vec{\nabla}\cdot\left(\kappa\,\vec{\nabla} n\right)$$

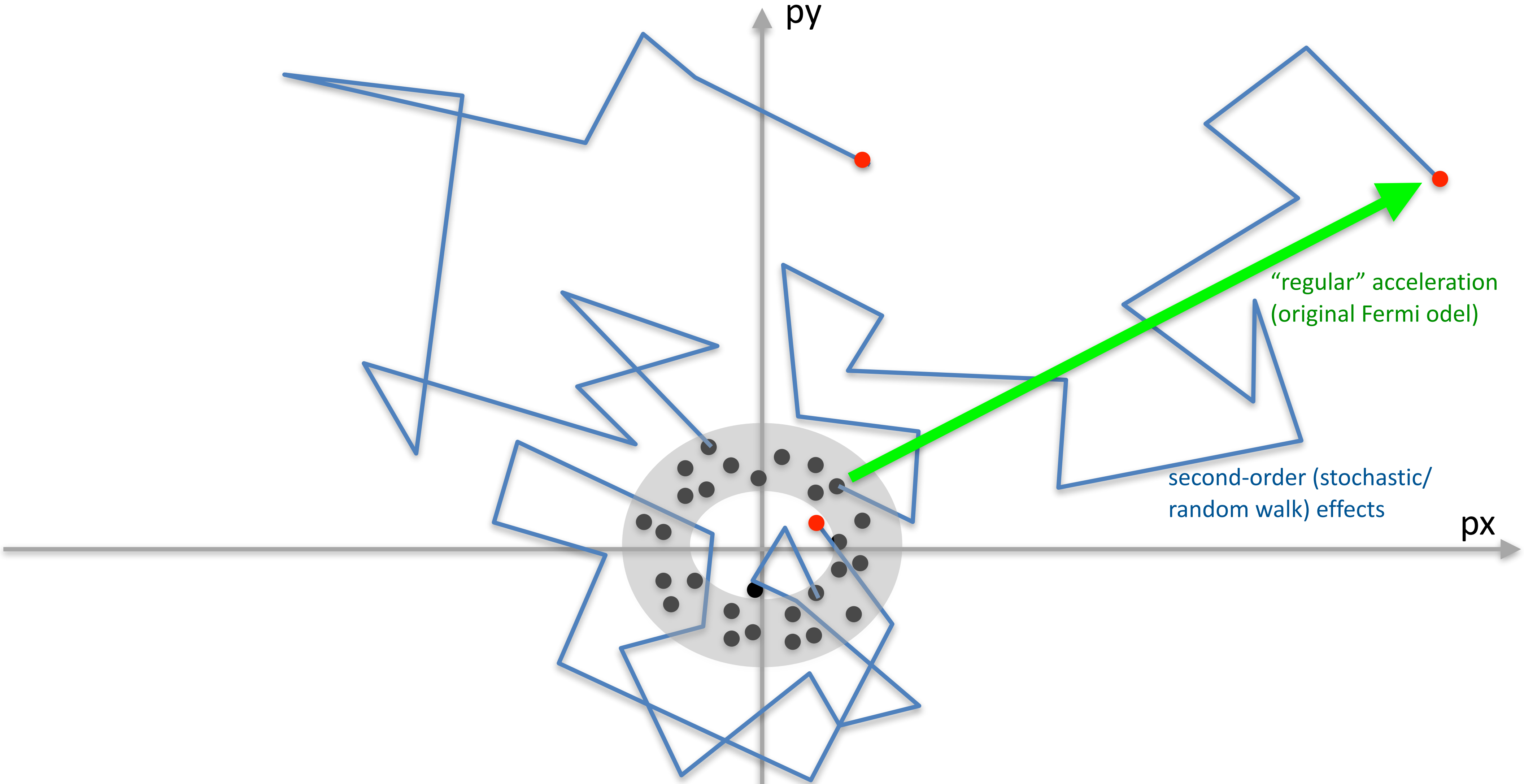
Remember that $\frac{D_p}{p} \equiv \frac{p}{\tau_{acc}}$, $E = pc$,

and note the Fermi's continuity equation:

"regular" acceleration term

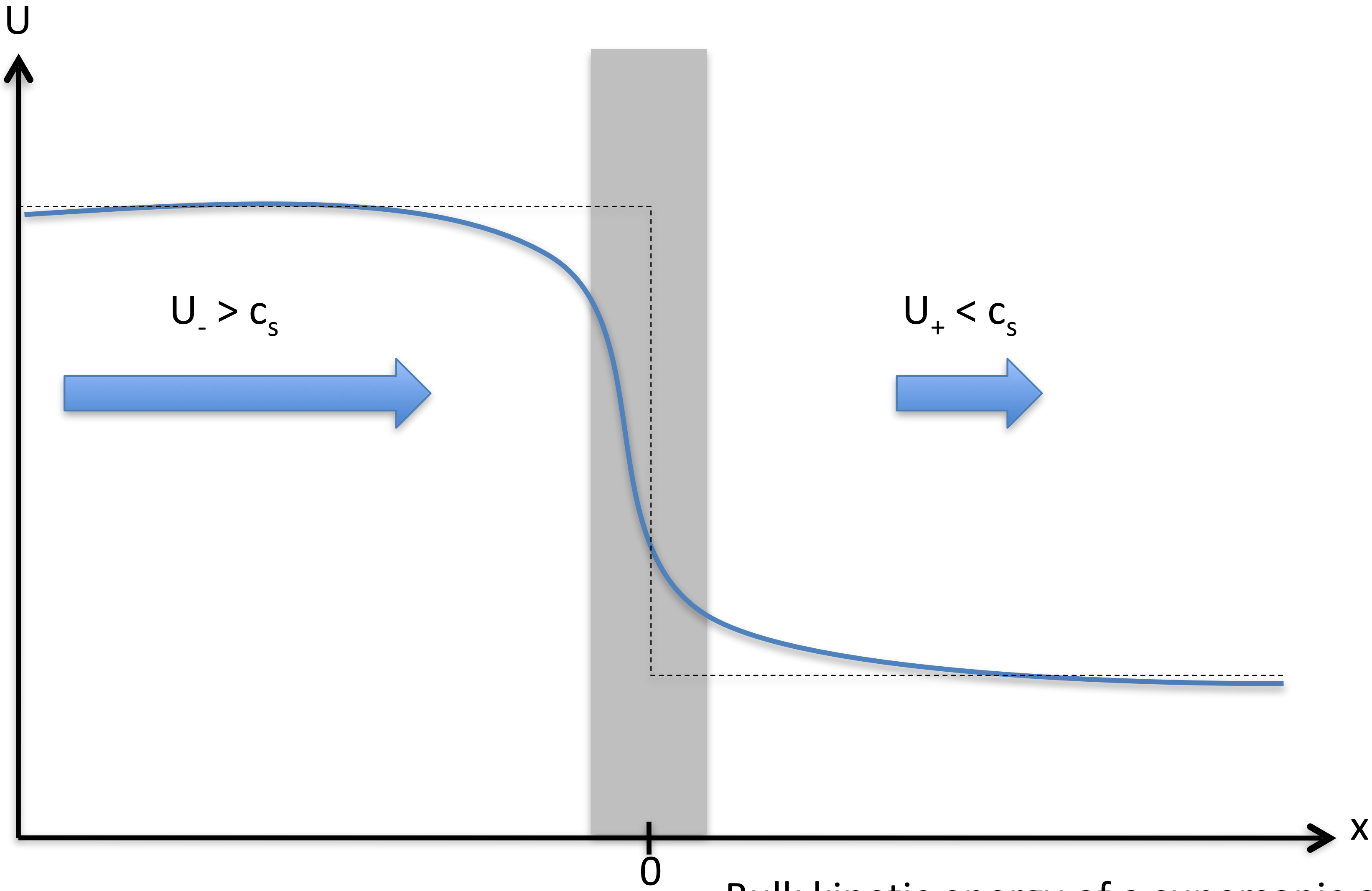
momentum diffusion approximated by a simple "escape" term

$$\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left(\frac{E}{\tau_{\rm acc}} n(E,t)\right) - \frac{n(E,t)}{\tau_{\rm esc}}$$



In its original formulation, the Fermi process is an approximate description of the momentum diffusion experienced by particles interacting with magnetic turbulence, with second-order (stochastic/ random walk) effects neglected.

Shocks



Bulk kinetic energy of a supersonic outflow is transferred to the plasma internal energy

Advection and Diffusion

What if we allow for compressions in the background plasma, particularly the formation of shocks?

A new term will arise in the transport equation, corresponding to particle advection in momentum space.

Advection in position space with velocity $\vec{U}$

$$\frac{\partial f}{\partial t} + \vec{U} \cdot \vec{\nabla} f =$$

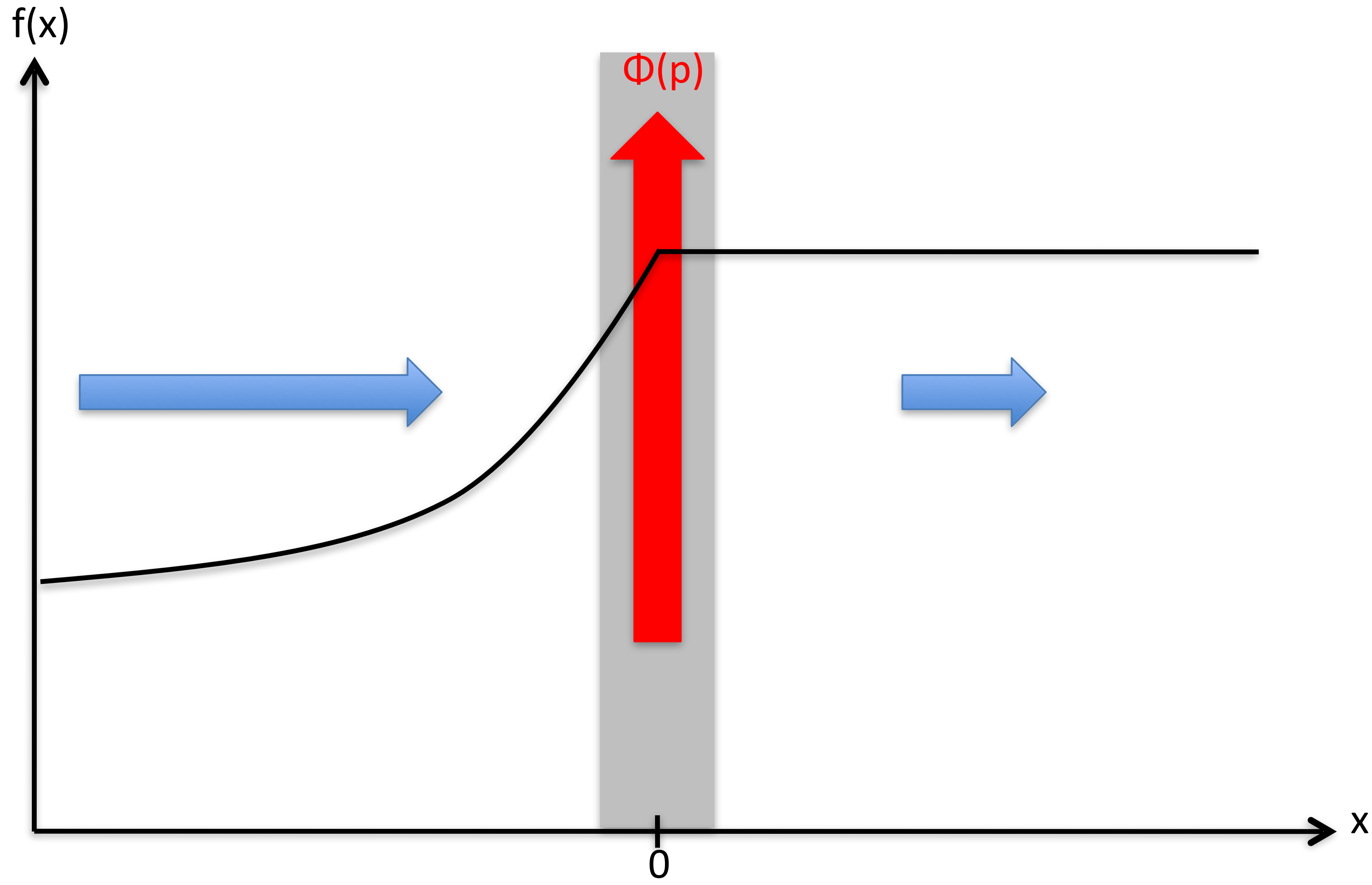
$$= \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_p \frac{\partial f}{\partial p} \right) + \vec{\nabla} \cdot \left(\kappa \vec{\nabla} f \right) + \frac{1}{3} \left(\vec{\nabla} \cdot \vec{U} \right) p \frac{\partial f}{\partial p}$$

Momentum diffusion Spatial diffusion Adiabatic compression

Advection in momentum space with velocitv

$$-\frac{1}{3} \left(\vec{\nabla} \cdot \vec{U} \right) p$$

Momentum Advection



reviews: Drury 1983, Blandford & Eichler 1987, Jones & Ellison 1991

At the Shock Front

Let's neglect momentum diffusion for simplicity *(which could be justified only if shock velocity is much larger than velocities of turbulent modes)*

$$\frac{\partial f}{\partial t} + \vec{U} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \left(\kappa \vec{\nabla} f \right) + \frac{1}{3} \left(\vec{\nabla} \cdot \vec{U} \right) p \frac{\partial f}{\partial p}$$

… and make the problem 1-dimensional:

$$U(x) = \begin{cases} U_- & \text{for} \quad x < 0 \\ U_+ & \text{for} \quad x \geq 0 \end{cases}$$

$$-\vec{\nabla} \cdot \vec{U} = (U_- - U_+)\, \delta(x)$$

Compression only at the shock front, i.e. at $x = 0$

Momentum flux

$$\Phi(p) = \int d^3x\, 4\pi\, p^2 \left[-\frac{1}{3} (\vec{\nabla} \cdot \vec{U})\, p \right] f(p) = \frac{4\pi\, p^3}{3} (U_- - U_+)\, f_0(p)$$

at the shock front

At the Shock Front

$$\partial_t f + U\,\partial_x f - \partial_x\left(\kappa\,\partial_x f\right) - \frac{1}{3}(\partial_x U)\,p\partial_p f = 0$$

let's now integrate each term from $-\epsilon$ to $+\epsilon$ along x-axis, keeping in mind that $U(x) = U_+ + (U_- - U_+)\,\Theta[-x]$, using Heaviside Theta, and hence $\partial_x U = -\,(U_- - U_+)\;\delta[x]$, using Dirac Delta. The first term is

$$\int_{-\epsilon}^{+\epsilon} dx\,\partial_t f \simeq \left(\Delta x|_{0+} + \Delta x|_{0-}\right)\;\partial_t f_0 \simeq \left(\ell_+ + \ell_-\right)\;\partial_t f_0 \simeq \left(\frac{\kappa_+}{U_+} + \frac{\kappa_-}{U_-}\right)\;\partial_t f_0$$

for the diffusion lengths in the upstream and downstream $\ell_\pm \simeq \kappa_\pm / U_\pm$. The last term is simply

$$\int_{-\epsilon}^{+\epsilon} dx\,\frac{1}{3}(\partial_x U)\,p\partial_p f = -\frac{1}{3}(U_- - U_+)\;p\partial_p f_0$$

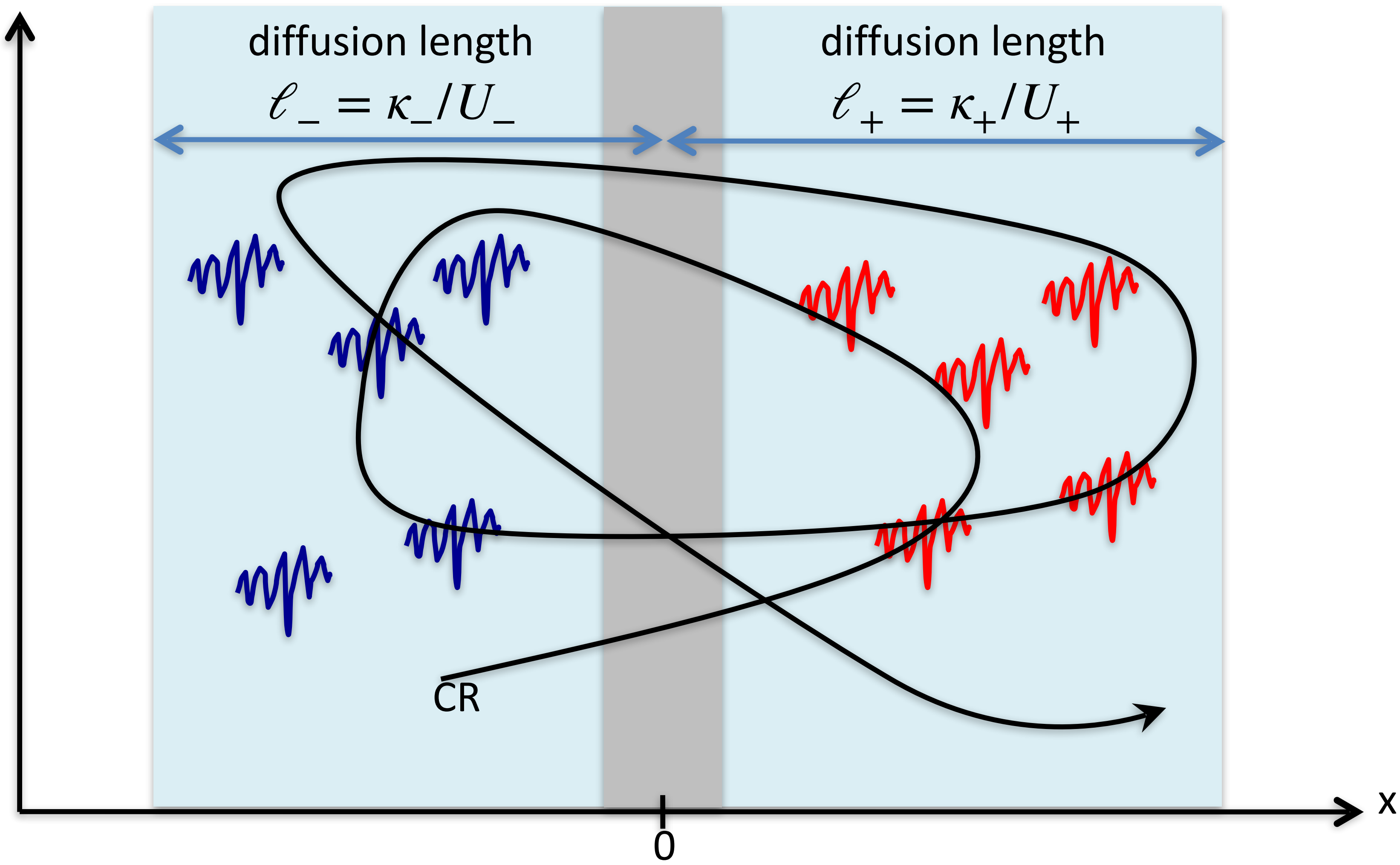
The two middle terms, meanwhile, are

$$\int_{-\epsilon}^{+\epsilon} dx\,\left[U\,\partial_x f - \partial_x\left(\kappa\,\partial_x f\right)\right] = \int_{-\epsilon}^{+\epsilon} dx\,\left[\partial_x\left(Uf - \kappa\,\partial_x f\right) - (\partial_x U)\,f\right] = \left[Uf - \kappa\,\partial_x f\right]_{-\epsilon}^{+\epsilon} - (U_- - U_+)f_0 =$$

$$= (U_+ f_+ - U_- f_-) - \left[\kappa\,\partial_x f\right]_{-\epsilon}^{+\epsilon} - (U_- - U_+)f_0 \simeq -\left[\kappa\,\partial_x f\right]_{-\epsilon}^{+\epsilon} \simeq +\kappa\,\partial_x f\,|_{-\epsilon} \simeq \frac{\kappa_-}{\ell_-} f_0 = U_- f_0$$

where we assumed continuity of f across the shock for small ϵ, namely $f_- \simeq f_+ \simeq f_0$, and the fact that in the downstream f is almost constant, so that $\partial_x f\,|^{+\epsilon} \simeq 0$.

It Still Is a Fermi Process!



Acceleration due to converging scattering centers which are advected with the flow; particles are isotropized at both sides of a shock, and each transformation upstream -> downstream -> upstream results in the energy gain.

Diffusive Shock Acceleration

$$\left(\frac{\kappa_-}{U_-} + \frac{\kappa_+}{U_+}\right)\frac{\partial f}{\partial t} + \frac{U_- - U_+}{3}\, p\, \frac{\partial f}{\partial p} = -U_-\, f$$

All in all, the resulting equation for the particle distribution at the shock front ($f = f_0$) can be solved straightforwardly using the method of characteristics.

$$\frac{dp}{dt} = \frac{U_- - U_+}{3}\, p \left(\frac{\kappa_-}{U_-} + \frac{\kappa_+}{U_+}\right)^{-1}$$

$$\frac{df}{dp} = -3\frac{U_-}{U_- - U_+}\, p^{-1}\, f$$

$$\tau_{\rm acc} = \frac{p}{dp/dt} = 3\left(\frac{\kappa_-}{U_-} + \frac{\kappa_+}{U_+}\right)(U_- - U_+)^{-1}$$

$$f(p) \propto p^{-\sigma} \quad \text{where} \quad \sigma = \frac{3R}{R-1}$$

universal spectrum!

$$R = \frac{U_-}{U_+}$$

Universal spectral index

since for a strong shock R = $U_-/U_+ \to 4$, then

$$\Rightarrow \quad f(p) \propto p^{-4} \quad \text{i.e.} \quad n(E) = f(p)\, 4\pi p^2 dp/dE\,|_{p=E/c} \propto E^{-2}$$