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An introduction to the theory of diffusive shock acceleration of energetic particles in tenuous plasmas

L O'C Drury

Max-Planck-Institut für Kernphysik, Postfach 103980, D6900 Heidelberg-1, West Germany

Abstract

The central idea of diffusive shock acceleration is presented from microscopic and macroscopic viewpoints; applied to reactionless test particles in a steady plane shock the mechanism is shown to produce a power law spectrum in momentum with a slope which, to lowest order in the ratio of plasma to particle speed, depends only on the compression in the shock. The associated time scale is found (also by a macroscopic and a microscopic method) and the problems of spherical shocks, as exemplified by a point explosion and a stellar-wind terminator, are treated by singular perturbation theory. The effect of including the particle reaction is then studied. It is shown that if the scattering is due to resonant waves these can rapidly grow with unknown consequences. The possible steady modified shock structures are classified and generalised Rankine–Hugoniot conditions found. Modifications of the spectrum are discussed on the basis of an exact, if rather artificial, solution, a high-energy asymptotic expansion and a perturbation expansion due to Blandford. It is pointed out that no steady solution can exist for very strong shocks; the possible time dependence is briefly discussed.

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1. Introduction

The subject of this review is a mechanism, basically simple, whereby shock waves in tenuous ionised media such as fill interstellar and interplanetary space can accelerate energetic charged particles. While of interest in itself the idea's chief attraction is that it may be a first step towards understanding the universal acceleration mechanism long suspected by workers in high-energy astrophysics to exist and to be responsible for the galactic cosmic rays and the energetic electrons inferred in many non-thermal radio sources (the main evidence is that all the particle energy spectra seem to be power laws with very similar indices). As sometimes happens in the history of Science the concept appears to have been 'in the air' and was independently and almost contemporaneously published by Krymsky (1977), Axford *et al* (1977), Bell (1978a, b) and Blandford and Ostriker (1978). Even prior to these four seminal papers which all, but with interesting variations, describe the same basic process of diffusive shock acceleration (also known as first-order or regular Fermi acceleration by shock waves) certain papers clearly foreshadowed the concept; thus Hoyle (1960), without specifying a mechanism, postulated that shocks could efficiently accelerate particles, Parker (1958) and Hudson (1965, 1967) attempted to obtain a first-order Fermi mechanism based on pairs of converging shocks and most notably Schatzman (1963) constructed a theory very similar to diffusive shock acceleration but based on perpendicular shocks. The idea is also implicit in the papers of Jokipii (1966, 1968), Fisk (1971) and Scholer and Morfill (1975).

In the six years since its first publication the concept has been considerably elaborated and is already the subject of reviews by Blandford (1979), Axford (1980, 1981a, b) and Toptyghin (1980). This review deals exclusively with the theoretical aspects of the mechanism; the relevant observational data as well as much theoretical background can be found in the monograph by Longair (1981) and in the volumes of conference proceedings containing Blandford's (1979) and Axford's (1980) reviews. Virtually all work in this area has been reported (and criticised) at the biannual international cosmic-ray conferences (the last three were at Plovdiv in 1977, Kyoto in 1979 and Paris in 1981); the published proceedings are an invaluable guide to current research and contain discussions of the mechanism's application to the acceleration of the galactic cosmic rays (cf in particular the incisive criticism by Ginzburg and Ptuskin (1981)). The theses of Achterberg (1981) and Ellison (1981) are also useful sources.

A shock wave, or briefly a shock, can be generally defined as a transition layer which propagates through a plasma and changes its state. The physically important case, and the only one considered here, is where the shock compresses the plasma and some of the kinetic energy of the incoming plasma is transferred to internal degrees of freedom of the downstream plasma. The thickness of the transition layer (the shock front) is determined by the physical process responsible for this energy conversion. In an ordinary gas shock the energy is transferred by two-body collisions to the random thermal motion of the gas molecules and the thickness is of the order of a few collisional mean free paths. However, in tenuous plasmas collisions are rare and the energy transfer proceeds through collective electromagnetic effects; thus the thickness of these collisionless shocks is of the order of the gyroradius of a thermal ion (or the Debye length if electrostatic effects are important). An energetic charged particle is one with sufficient momentum not to resonate with this electromagnetic turbulence in the shock front and thus see the shock essentially as a discontinuity (whereas the 'thermal' particles interact strongly and are 'heated'). The general properties of shocks are discussed in the books by Zeldovich and Raizer (1966), Whitham (1974) and Lighthill (1978); the standard reference on collisionless shocks is the monograph by Tidman and Krall (1971).

The notation and nomenclature used in this review is, in general, standard. The Dirac distribution is denoted as usual by δ and the Heaviside unit step function by H. The symbol $\wedge$ is used both for the cross product of two vectors (denoted by boldface type) and for the exterior (or wedge) product of differential forms. 'Cosmic ray' is often used as a convenient synonym for 'energetic charged particle' and the definition of pitch in (2.1) is non-standard. Momentum always means kinetic momentum (not the canonical momentum which contains an electromagnetic part) but this does not affect the use of Liouville's theorem.

2. Basic theory

This section begins with a simple heuristic derivation of the diffusive transport equation. The kinematic structure (i.e. space–time geometry) of the electromagnetic and velocity fields at a shock is then discussed and scatter-free acceleration briefly considered. The third, and major, part discusses as simply as possible the fundamental ideas of diffusive shock acceleration.

2.1. Transport of energetic charged particles

We begin with the elementary result that in a uniform magnetic field a freely moving charged particle follows a helical trajectory, the superposition of a uniform translation parallel to the field and a regular gyration perpendicular to the field. The particle's pitch, μ, is the cosine of the angle between its momentum vector $\boldsymbol{p}$ and the magnetic field $\boldsymbol{B}$:

$$\mu = \boldsymbol{p} \cdot \boldsymbol{B}/pB \tag{2.1}$$

and its parallel and perpendicular momenta are

$$\begin{aligned} p_{\parallel} &= \mu p \\ p_{\perp} &= (1-\mu^2)^{1/2} p. \end{aligned} \tag{2.2}$$

The gyroradius is

$$r_g = p_{\perp}/ZeB \tag{2.3}$$

where Ze is the particle's electric charge (in SI units; in Gaussian units the RHS must be multiplied by the speed of light).

Fortunately the real world is not as simple as first-year physics; cosmic magnetic fields are usually neither static nor uniform, they contain irregularities and variations which perturb the particles and lead to scattering. If we think of the simplest case, a homogeneous background medium permeated by a uniform magnetic field on which are imposed small static irregularities, then because the electric field is identically zero particles conserve their energy (and scalar momentum) on scattering but have their pitch changed (and also their gyrophase, but this is usually unimportant). If the scattering is 'sufficiently stochastic' (it would be interesting to be able to give a precise definition of this) the distribution function $F(\boldsymbol{p}, \boldsymbol{x}, t)$ or phase space density of the particles is kept close to isotropy, $F(\boldsymbol{p}, \boldsymbol{x}, t) \sim f(p, \boldsymbol{x}, t)$ where f is the isotropic part of F, and the particle transport can be described by the diffusion equation

$$\frac{\partial f}{\partial t} = \boldsymbol{\nabla}(\boldsymbol{\kappa}\boldsymbol{\nabla} f) \tag{2.4}$$

where $\boldsymbol{\kappa}$ is the (anisotropic) diffusion tensor. Much effort has gone into calculating κ in terms of the irregularity spectrum (see, for example, Völk (1973) or Skilling (1975) and references therein) but the general nature of the result is adequately shown by the following simple argument (cf Blandford (1979) and Longair (1981)).

The irregularities which are most effective at scattering a particle are those with length scales comparable to its gyroradius; the mean square field variation due to these is of the order of $kI(k)$ where $k \sim r_g^{-1}$ and $I(k)$ is the spatial power spectrum of the irregularities. The mean square variation in the field direction on the length scale r_g is thus

$$\varphi^2 \sim kI(k)/B_0^2 \tag{2.5}$$

where B_0 is the uniform mean field. Each gyroperiod the particle's pitch angle is changed a random amount of the order of φ and after N periods the accumulated change is of the order of $N^{1/2}\varphi$. This becomes of the order of unity, i.e. the particle 'forgets' its original pitch, when $N \sim \varphi^{-2} \sim B_0^2[kI(k)]^{-1}$ so that the mean free path along the field is $\lambda_\parallel \sim N r_g$ and the scattering frequency $\nu \sim V/\lambda_\parallel$. The diffusion coefficient parallel to the field is thus

$$\kappa_\parallel \sim \tfrac{1}{3}\lambda_\parallel^2 \nu \sim \tfrac{1}{3} r_g v B_0^2 [kI(k)]^{-1}. \tag{2.6}$$

Between scatterings a particle accumulates a displacement perpendicular to the mean field of the order of

$$\lambda_\perp \sim N^{1/2}\varphi r_g \sim r_g \tag{2.7}$$

so that the perpendicular diffusion coefficient is

$$\kappa_\perp \sim \tfrac{1}{3}\lambda_\perp^2 \nu \sim \tfrac{1}{3} r_g v/N. \tag{2.8}$$

Thus

$$\kappa_\parallel \kappa_\perp \sim \kappa_{\mathrm{B}}^2 \tag{2.9}$$

where $\kappa_{\mathrm{B}} = \frac{1}{3} r_g v$, the 'Bohm' diffusion coefficient, is essentially the smallest diffusion coefficient allowed by this model of particle transport (it corresponds to a totally random field on the scale r_g).

This assumes scattering by stationary magnetic irregularities (scattering centres) which is, of course, unrealistic. There are two types of scattering centre motion which must be considered. First, there can be large-scale motions of the background system which advect the centres with velocity $\boldsymbol{U}$. Clearly this requires us to replace the time derivative in (2.4) by a convective derivative:

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \tag{2.10}$$

and also, if the large-scale motion is convergent or divergent, we must add a term to represent adiabatic changes in the particle momenta. The form of this term can be deduced either by energy conservation or, more simply, by using Liouville's theorem; a convergence in position space must be matched by an equal divergence in momentum space. These heuristic arguments (similar to those used by Parker (1965); cf also Dolginov and Toptyghin (1966)) lead to the transport equation

$$\frac{\partial f}{\partial t}+\boldsymbol{U}\cdot\nabla f=\boldsymbol{\nabla}(\boldsymbol{\kappa}\boldsymbol{\nabla} f)+\tfrac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{U}p\frac{\partial f}{\partial p} \tag{2.11}$$

for diffusive transport in a medium where the scattering centres are fixed in the background.

There remains the second type of motion, that of the individual centres relative to the background. If this has a systematic component it should obviously be added to the background velocity $\boldsymbol{U}$ (the energetic particles cannot see the background directly, they can only sense the mean velocity of the scattering centres). The residual random component causes each particle to have its momentum changed by a random amount of the order of $\Delta p \sim pV/v$ at each scattering where V is the velocity of a centre. This gives rise to a diffusion in momentum space with coefficient

$$D\sim\tfrac{1}{3}(\Delta p)^2\nu\sim V^2p^2/q\kappa_\parallel \tag{2.12}$$

which represents classical second-order Fermi acceleration (Fermi 1949, 1954) so that the complete transport equation is

$$\frac{\partial f}{\partial t}+\boldsymbol{U}\cdot\nabla f=\boldsymbol{\nabla}(\boldsymbol{\kappa}\boldsymbol{\nabla} f)+\tfrac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{U}p\frac{\partial f}{\partial p}+\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2D\frac{\partial f}{\partial p}\right). \tag{2.13}$$

A formal derivation of essentially this equation is given in the paper by Skilling (1975). The velocity V is the propagation speed of a small disturbance in the plasma, thus under astrophysical conditions it will usually be of the order of the Alfvén speed and is often small enough to be neglected.

2.2. Shock kinematics and scatter-free acceleration

Let us leave, for the moment, the dynamical questions of scattering and transport and turn to the kinematical questions of the magnetic-field configuration at a shock and the particle trajectories in this field. At least locally we can treat the shock front as plane and the upstream and downstream media as uniform with velocities $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ and magnetic fields $\boldsymbol{B}_1$, $\boldsymbol{B}_2$. In almost all cases the high plasma conductivity will ensure that the electric field vanishes in the plasma rest frame (so that $\boldsymbol{E}_{1,2}=-\boldsymbol{U}_{1,2}\wedge\boldsymbol{B}_{1,2}$). Let us begin in an inertial frame moving with the upstream fluid and look at the velocity with which the point of intersection between a magnetic-field line and the shock front moves. There is a physically significant distinction between those shocks in which this velocity is subluminal and those in which it is greater than or equal to the speed of light; only in the first case can charged particles be reflected from the shock.

In the subluminal case, by boosting parallel to the field at the velocity of the intersection point, we find a proper Lorentz transformation which takes us to a frame in which $\boldsymbol{U}_1$ is parallel to $\boldsymbol{B}_1$, $\boldsymbol{E}_1=0$ and the shock front is stationary. We can then translate the origin to the shock front and rotate the coordinate system so that the upstream region is the half-space $x<0$ and $\boldsymbol{U}_1$ lies in the xy plane. The electromagnetic matching conditions imply that

$$\begin{aligned} E_{2y,z} &= E_{1y,z} = 0 \\ B_{2x} &= B_{1x} \neq 0 \end{aligned} \tag{2.14}$$

but $\boldsymbol{E}_2 \cdot \boldsymbol{B}_2 = 0 \Rightarrow E_{2x}B_{2x} = 0 \Rightarrow E_{2x} = 0$; thus $\boldsymbol{E}_2 = 0$ and $\boldsymbol{U}_2$ is also parallel to $\boldsymbol{B}_2$. In this way, if the shock is subluminal, we can construct an essentially unique frame in which the shock is stationary and the electric field vanishes everywhere (figure 1(*a*)).

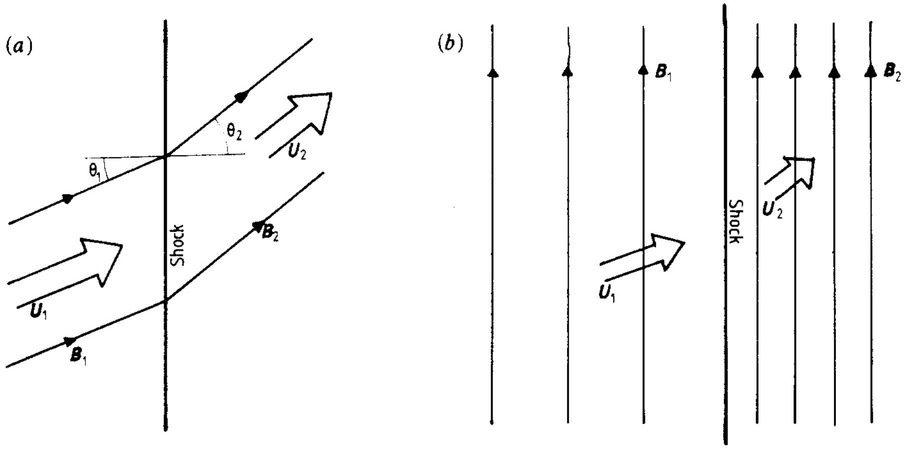

Figure 1. The simplified normal forms of the magnetic and velocity fields at a steady plane shock where the intersection point between a magnetic-field line and the front moves (*a*) subluminally and (*b*) superluminally.

The kinematic structure of the shock is contained in the angles θ_1, θ_2 between $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and the shock normal and the compression ratio $r = U_{1x}/U_{2x}$. If $\theta_1 = \theta_2 = 0$ the shock is said to be parallel, else oblique. The dynamical constraints imposed by mass momentum and energy conservation (which give the MHD Rankine–Hugoniot conditions: see de Hoffmann and Teller (1950) and Lüst (1955a, b)) show that, except in very unusual circumstances, $\boldsymbol{U}_2$ and $\boldsymbol{B}_2$ lie in the xy plane and relate θ_2 and r to the upstream dynamical parameters. The general formulae are quite complicated, but the special case of a strong shock (where the kinetic energy density and kinetic pressure dominate upstream) is simple and instructive. From mass conservation

$$\rho_1 U_{1x} = \rho_2 U_{2x} = A \tag{2.15}$$

where ρ is the mass density, momentum conservation gives

$$\begin{aligned} U_{1y} &= U_{2y} \\ AU_{1x} &= AU_{2x} + P \end{aligned} \tag{2.16}$$

where P is the downstream pressure and energy conservation gives

$$\begin{aligned} &\tfrac{1}{2}AU_{1x}^2 = \tfrac{1}{2}AU_{2x}^2 + U_{2x}(E+P) \\ &\Rightarrow \tfrac{1}{2}A(U_{1x} - U_{2x})(U_{1x} + U_{2x}) = U_{2x}(E+P) \\ &\Rightarrow P(U_{1x} + U_{2x}) = 2U_{2x}(E+P) \\ &\Rightarrow r = 1 + \frac{2E}{P} \end{aligned} \tag{2.17}$$

where E is the downstream internal energy density. For a monatomic non-relativistic gas $P=\frac{2}{3}E$ and $r=4$; for a relativistic gas $P=\frac{1}{3}E$ and $r=7$. From $U_{1x}=rU_{2x}$ and $U_{1y}=U_{2y}$ we deduce that $r\tan\theta_1=\tan\theta_2$.

But what if the point of intersection moves superluminally at velocity $v>c$? Then a boost along the field at c^2/v is a proper Lorentz transformation to a frame where this velocity is infinite, i.e. where the entire field line intersects the shock front simultaneously. In this frame the magnetic field upstream is perpendicular to the shock normal but the shock is not stationary; however, by boosting along the shock normal at the shock speed we can find a frame where the field $\boldsymbol{B}_1$ (and hence also $\boldsymbol{B}_2$) is still perpendicular to the shock normal and the shock is stationary. The incoming fluid flows in at velocity $\boldsymbol{U}_1$ and at an angle θ_1 to the shock normal and exists at $\boldsymbol{U}_2$, θ_2: the upstream electric field, $-\boldsymbol{U}_1\wedge\boldsymbol{B}_1$, is of magnitude $U_1B_1\cos\theta$; matching to the downstream field shows that $\boldsymbol{E}_1=\boldsymbol{E}_2\neq 0$ and $B_2=rB_1$ (figure 1(b)).

Thus in both the subluminal and superluminal cases we can find certain frames in which the electromagnetic field has an especially simple form. The luminal case does not admit such simplification, but is exceptional and physically should probably be regarded as a degenerate form of the superluminal case. We now examine energetic particle trajectories in these fields, a problem considered for the case of a purely perpendicular shock by Dorman and Freidman (1959), Shabanskii (1961) and Schatzman (1963) and for the general case by Hudson (1965), Alekseev and Kropotkin (1970) and many subsequent workers.

Let us begin by looking at a superluminal shock; then the fields reduce to an electric field

$$\boldsymbol{E}=-U_{1x}B_1\boldsymbol{e}_y=-U_{2x}B_2\boldsymbol{e}_y \tag{2.18}$$

perpendicular to the magnetic field

$$\boldsymbol{B}=\begin{cases} B_1\boldsymbol{e}_y & x<0 \\ rB_1\boldsymbol{e}_y & x>0. \end{cases} \tag{2.19}$$

The trajectory of an energetic particle in the upstream region consists of the usual helical motion in the $\boldsymbol{B}$ field superimposed on an '$\boldsymbol{E}\wedge\boldsymbol{B}$' drift towards the shock at velocity U_{1x}. As the particle (or rather its guiding centre) drifts through the shock the smaller downstream gyroradius causes a drift parallel to $\boldsymbol{E}$ which increases its momentum. This effect can be rather simply calculated in the limit $v\gg U_{1x}$ by applying Liouville's theorem. The momentum gained by a particle is then almost independent of its gyrophase so that

$$2\pi p_{\perp 2}\,\mathrm{d}p_{\perp 2}\wedge\mathrm{d}p_{\parallel 2}\wedge\mathrm{d}x_2\wedge\mathrm{d}y_2\wedge\mathrm{d}z_2=2\pi p_{\perp 1}\,\mathrm{d}p_{\perp 1}\wedge\mathrm{d}p_{\parallel 1}\wedge\mathrm{d}x_1\wedge\mathrm{d}y_1\wedge\mathrm{d}z_1. \tag{2.20}$$

But clearly $\mathrm{d}x_2\wedge\mathrm{d}y_2\wedge\mathrm{d}z_2=(1/r)\,\mathrm{d}x_1\wedge\mathrm{d}y_1\wedge\mathrm{d}z_1$ and $p_{\parallel 2}=p_{\parallel 1}$. Thus

$$p_{\perp 2}\,\mathrm{d}p_{\perp 2}=rp_{\perp 1}\,\mathrm{d}p_{\perp 1} \tag{2.21}$$

and integrating

$$p_{\perp 2}^2=rp_{\perp 1}^2+\text{constant}. \tag{2.22}$$

The constant can be found by noting that $p_{\perp 1}=0$ implies (in this limit, $v\gg U_{1x}$)$p_{\perp 2}=0$; it follows that

$$\frac{p_{\perp 2}^2}{B_2}=\frac{p_{\perp 1}^2}{B_1} \tag{2.23}$$

i.e. the particle approximately conserves its first magnetic moment as it would have (and for exactly the same reason, namely approximate gyrophase independence) had the change in the magnetic field been adiabatic rather than abrupt.

We see that the perpendicular momentum of an energetic particle can be increased by a factor of the square root of the compression ratio. With r in the range 4–7 this is not much acceleration. Furthermore a subsequent expansion of the medium to its original density annuls the effect.

We now turn to the subluminal case. Here we can find a frame where the electromagnetic field reduces to a purely magnetic field and in which, therefore, particles must conserve their energy. Whereas in the superluminal case all particles were forced to go through the shock, and had in consequence to increase their energy to find enough downstream phase space, in this case, because they are not allowed to increase their energy, there is not enough phase space for all particles to be transmitted and some must be reflected. Considering particles of momentum p, upstream pitch μ_1 and gyrophase α_1, if these are transmitted with pitch μ_2 and gyrophase α_2 then Liouville's theorem states that

$$p^2 \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1 \wedge \mathrm{d}x_1 = p^2 \, \mathrm{d}\mu_2 \wedge \mathrm{d}\alpha_2 \wedge \mathrm{d}x_2$$

but

$$\mathrm{d}x_{1,2} = \mu_{1,2} v \, \mathrm{d}t \cos\theta_{1,2} \tag{2.24}$$

so that

$$\cos\theta_1 \mu_1 \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1 = \cos\theta_2 \mu_2 \, \mathrm{d}\mu_2 \wedge \mathrm{d}\alpha_2 \tag{2.25}$$

or

$$\frac{1}{B_1} \mu_1 \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1 = \frac{1}{B_2} \mu_2 \, \mathrm{d}\mu_2 \wedge \mathrm{d}\alpha_2 \tag{2.26}$$

and there is a similar formula for reflected particles. The assumption of approximate gyrophase invariance again implies approximate conservation of the magnetic moment, as verified by numerical simulations (e.g. Terasawa 1979).

If all particles incident on the shock from upstream were transmitted, then

$$\int_0^{2\pi} \int_0^1 \frac{1}{B_1} \mu_1 \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1 \leqslant \int_0^{2\pi} \int_0^1 \frac{1}{B_2} \mu_2 \, \mathrm{d}\mu_2 \wedge \mathrm{d}\alpha_2 \Rightarrow \frac{1}{B_1} \leqslant \frac{1}{B_2} \Rightarrow B_2 \leqslant B_1 \tag{2.27}$$

so that particles must be reflected if the downstream field is stronger than the upstream. There is, however, no need for particles incident from downstream to be reflected and indeed they are not as shown by Hudson (1965). An interesting and important implication of these formulae is that, if the upstream and downstream distributions are isotropic and have the same density, the combined distribution of particles transmitted through and reflected from the shock is also isotropic and has the same uniform density. This clearly follows on thermodynamic grounds; if the distribution were anisotropic this would imply a violation of the second law.

As far as acceleration is concerned the subluminal case looks even worse than the superluminal because we have perfect conservation of particle energy. However, this is to some extent an artefact of the frame we are using; with respect to other reference systems, in particular that in which the incoming fluid flows parallel to the shock normal, the reflected particles are accelerated. The effect can be quite substantial if the transformation to the $\boldsymbol{E}=0$ frame involves 'large' velocities, i.e. for quasi-perpendicular shocks, but is not enormous; the maximum kinetic energy gain of a non-relativistic particle is

$$\Delta T \sim 4T(r-1) \tag{2.28}$$

and for a relativistic particle the momentum gain is only

$$\Delta p \sim 2p(r-1) \tag{2.29}$$

(Toptyghin 1980). Of course, these energy changes can be interpreted in terms of particle drifts at the shock parallel to the $\boldsymbol{U} \wedge \boldsymbol{B}$ electric field (cf Webb *et al* 1983).

2.3. *Diffusive acceleration at shocks*

We have seen in the last subsection that particles can gain energy by interacting once with a shock front. However, this effect, even in the most favourable cases, can only increase the energy of a particle by a moderate amount; furthermore, as the process is purely kinematic and reversible the spectrum of the accelerated particles is essentially the pre-acceleration spectrum shifted in energy. This situation is completely changed when diffusive effects are included. The number of times a particle interacts with the shock then becomes a random variable and some particles, by interacting many times with the shock, achieve very high energies. This stochastic aspect also means that the process destroys information (equivalently entropy is generated, not only in the background plasma, but also in the energetic particle distribution) so that the spectrum of the accelerated particles can be relatively independent of the details of the pre-acceleration spectrum. Studies of the simplest version of this effect, the steady diffusive acceleration of test particles at a one-dimensional parallel shock, have followed two approaches: a microscopic approach in which the energy changes and histories of individual particles are studied and a macroscopic approach which uses the formalism of the distribution function and its associated transport equation.

2.3.1. The macroscopic approach. In this version (historically the first (Krymsky 1977, Axford *et al* 1977, Blandford and Ostriker 1978)) it is assumed that the velocity of the particles is very much greater than that of the fluid and that scattering of the particles keeps the distribution function almost isotropic in the fluid frame (so that the diffusion approximation is valid), that this scattering is elastic, and that the scatterers behave as if they had an infinite effective mass (the particle energy in the local fluid frame does not change on scattering). The shock front is taken to be an infinite plane separating uniform upstream and downstream media and perpendicular to the uniform magnetic field (i.e. the field lines are parallel to the shock normal). From the standard theory of shock waves we know that there exists an inertial frame in which the shock front is stationary and occupies the y, z plane and in which the fluid flows parallel to the x axis. In this frame all quantities depend spatially only on x (it is in this sense that the problem is one-dimensional) and the flow velocity is steady and given by

$$U(x)=\begin{cases} U_1 & x<0 \\ U_2 & x>0 \end{cases} \tag{2.30}$$

where U_1 and U_2 are the, constant, upstream and downstream velocities. The transport equation for the isotropic part of the energetic particle phase space density as seen in the local fluid frame, $f(t, x, p)$, is then

$$\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\kappa(x,p)\frac{\partial f}{\partial x}\right)+\frac{1}{3}\frac{\partial U}{\partial x}p\frac{\partial f}{\partial p} \tag{2.31}$$

where $\kappa(x, p)$ is the spatial diffusion coefficient for particles of momentum p along the field lines.

Having chosen a frame such that the background plasma flow is steady it is natural to look for steady solutions of (2.31). Upstream and downstream this reduces to

$$U\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\kappa\frac{\partial f}{\partial x} \tag{2.32}$$

where U equals U_1 or U_2 and is constant. The general solution is easily found to be

$$f(x,p)=g_1(p)\exp\left(\int_0^x\frac{U(x')}{\kappa(x',p)}\,\mathrm{d}x'\right)+g_2(p) \tag{2.33}$$

where g_1 and g_2 are arbitrary functions of p. If we now impose the natural boundary conditions that f tend to some given distribution f_1 far upstream, i.e. $f(x, p)\to f_1(p)$ as $x\to-\infty$, and remain finite downstream, i.e. $|f(x, p)|<\infty$ as $x\to\infty$, then

$$f(x,p)=\begin{cases} f_1(p)+g_1(p)\exp\left(\int_0^x\frac{U\,\mathrm{d}x'}{\kappa}\right) & x<0\\ g_2(p)=f_2(p) & x\geqslant 0\end{cases} \tag{2.34}$$

as long as $\int_0^x \mathrm{d}x'/\kappa(x', p)\to\pm\infty$ when $x\to\pm\infty$. (If this condition is not satisfied stricter boundary conditions are required.) The physical content here is simply that in the diffusion-advection equation (2.32) a steady state can be achieved by balancing the diffusive flux against the advected flux only in the upstream half-space.

To relate the two unknown functions f_2 and g_1 to f_1 we need two matching conditions at the shock. The key here is that a parallel shock does nothing locally to energetic particles; there is no discontinuity in the magnetic field and the particles simply continue their helical paths across the front. Thus the complete momentum space distribution function, as measured in the shock frame, must be continuous across the front.

In the diffusion approximation the angular dependence in the fluid frame of the complete distribution function, $F(x, p, \mu)$, can be obtained by expanding F in a series of Legendre polynomials, truncating after two terms and setting the anisotropic part proportional to the gradient of the isotropic part; thus

$$F(x,p,\mu)\sim f(x,p)-\mu\lambda\frac{\partial}{\partial x}f(x,p) \tag{2.35}$$

where λ is the 'scattering mean free path' related to κ by $\kappa=\lambda v/3$ where v is the particle velocity. Inserting expression (2.34) for the isotropic part and evaluating just upstream and just downstream from the shock gives

$$F(x,p,\mu)\sim\begin{cases} f_1+g_1-\mu\dfrac{3U_1}{v}g_1 & x=0-\\ f_2 & x=0+.\end{cases} \tag{2.36}$$

In this expression p (but not x) is measured in the local fluid frame; on transforming to the shock frame, which moves at velocity $-U$ relative to the fluid, to order U/v

the particle momenta transform as

$$p \mapsto p' = p\left(1 - \mu \frac{U}{v}\right) \tag{2.37}$$

and in consequence, phase space densities being invariant,

$$F(x, p, \mu) \mapsto F(x, p', \mu') = \begin{cases} f_1 + g_1 - \mu\left(\dfrac{U_1}{v} p \dfrac{\partial}{\partial p}(f_1 + g_1) + \dfrac{3U_1}{v} g_1\right) & x = 0- \\ f_2 - \mu\left(\dfrac{U_2}{v} p \dfrac{\partial}{\partial p} f_2\right) & x = 0+. \end{cases} \tag{2.38}$$

Equating the isotropic and anisotropic parts of the upstream and downstream distributions then gives the two required matching conditions, namely

$$f_1 + g_1 = f_2 \tag{2.39}$$

and

$$U_1 p \frac{\partial}{\partial p}(f_1 + g_1) + 3U_1 g_1 = U_2 p \frac{\partial}{\partial p} f_2. \tag{2.40}$$

These are usually interpreted as expressing continuity of the particle number density and particle streaming at all momenta.

It is interesting to note that these matching conditions are in a sense already contained within the transport equation (2.31) itself; if we require this to hold, not only upstream and downstream from the shock, but also at the front itself, then we recover exactly the above conditions. Of course, as U is discontinuous at the shock, and perhaps κ also, we cannot require f to satisfy (2.31) there in the usual sense; we can, however, require that it be what is technically called a weak solution. Roughly speaking, this means that if we multiply both sides of the equation by a smooth test function, integrate across the shock from $-\epsilon$ to ϵ and interpret all ambiguous terms by formally applying integration by parts, then on taking the limit $\epsilon \to 0$ the result should be an identity for all test functions. Thus multiplying by $\int_0^x \mathrm{d}x'/\kappa$ and noting that the worst singularity allowed by the representation (2.34) is a simple discontinuity in f at $x = 0$, after some manipulation and estimation we derive

$$[f]_{-\epsilon}^{+\epsilon} = \mathrm{O}(\epsilon) \tag{2.41}$$

from which it follows that f is in fact continuous at $x = 0$. If we now use 1 as a test function we get

$$\left[\kappa \frac{\partial f}{\partial x}\right]_{-\epsilon}^{+\epsilon} + \tfrac{1}{3}[U]_{-\epsilon}^{+\epsilon} p \frac{\partial f}{\partial p} = \mathrm{O}(\epsilon) \tag{2.42}$$

which gives the second matching condition and fixes the discontinuity in the derivative of f. One can then show that if these conditions are satisfied the integral with any other test function is also $\mathrm{O}(\epsilon)$. That we are able to derive the boundary conditions from the transport equation in this way is simply another expression of the fact that at the microscopic level a parallel shock does nothing to energetic particles and should not be given any deeper significance.

Eliminating g_1 between (2.39) and (2.40) and introducing $r = U_1/U_2$, the shock compression ratio, we obtain

$$(r-1) p \frac{\partial f_2}{\partial p} = 3r(f_1 - f_2) \tag{2.43}$$

as the required equation relating the distribution function f_1 far upstream to that downstream, f_2. Setting $a = 3r(r-1)^{-1}$ the solution is easily found by elementary means to be

$$f_2(p) = ap^{-a} \int_0^p p'^{a-1} f_1(p')\, \mathrm{d}p' + bp^{-a} \tag{2.44}$$

where b is an arbitrary constant of integration. The first term can be interpreted as the spectrum of particles advected into the shock from upstream and there accelerated; the second as representing particles injected from the thermal background plasma into the same acceleration process. Ignoring for the moment the second term, the first term can be written

$$f_2(p) = \int_0^\infty a(p/p')^{-a} f_1(p') H(p-p')\, \mathrm{d}p'/p' \tag{2.45}$$

so that for those particles which are advected through the shock the downstream spectrum is obtained from the upstream spectrum by convolving with a truncated power law.

The significance of this is that if the upstream spectrum is softer than a power law spectrum with slope a, then the downstream spectrum has as asymptote at high momenta a power law spectrum of slope a regardless of the detailed form of the incoming spectrum. Furthermore, shocks in fluids with a ratio of specific heats γ have a compression ratio

$$r = \frac{\gamma+1}{\gamma-1+M^{-2}} \tag{2.46}$$

where M is the shock Mach number so that for the standard case of a strong shock in a non-relativistic plasma, $\gamma = 5/3$, $M \to \infty$, we have $r \to 4-$ and $a \to 4+$. This is encouragingly close to the index of 4.3 inferred for the source of the galactic cosmic rays.

2.3.2. Microscopic derivation. The above macroscopic analysis is straightforward; however it appears to lack physical content (essentially because this has all gone into deriving the transport equation); at the end of a formal analysis we find a downstream spectrum which contains accelerated particles, but how were they accelerated and why should they have a power law spectrum? The advantage of the alternative microscopic derivation (Bell 1978a, see also Peacock 1981, Michel 1981) is that it reveals the basic physical processes responsible for the acceleration.

Let us consider an energetic particle which crosses the shock front from upstream to downstream and ask the question: what is its probability of never returning to the shock, that is of escaping downstream to ∞? Denote the number density of particles having velocity v relative to the local fluid frame by n; then from the diffusion model we know that n is constant downstream. The flux of such particles escaping to ∞ is nU_2 whereas the flux entering the downstream region by crossing the front from upstream to downstream is $\int_0^1 \mu v n\, \mathrm{d}\mu/2 = nv/4$ (assuming an almost isotropic distribution; of the n particles one-half are moving to the right with an average projected velocity of one-half v). Thus the probability of not returning must be $nU_2(nv/4)^{-1} = 4U_2/v$ which by assumption is small, i.e. almost all particles cross the shock many times.

We now ask: what is the average change in a particle's momentum with respect to the local fluid frame when it crosses the shock? If the particle has momentum p, velocity v and pitch μ in the upstream fluid frame, then in the shock frame its momentum is $p(1+\mu U_1/v)$. This is unchanged on crossing the shock, and thus relative to the downstream frame its momentum is $p[1+\mu(U_1-U_2)/v]$. Note that as we are only working to order U/v we can ignore the small changes in pitch angle as we go from frame to frame and also the anisotropy of the distribution when averaging. Thus the average change in momentum is

$$\langle \Delta p\rangle = p\int_0^1 [\mu(U_1-U_2)/v]2\mu \, d\mu$$
$$= \tfrac{2}{3}p(U_1-U_2)/v \tag{2.47}$$

(assuming isotropy, the probability of crossing the shock at an angle θ is proportional to $\cos\theta$, hence the weighting factor of 2μ). On re-crossing the shock in the other direction U_1 and U_2 are interchanged, but μ runs from 0 to -1 so that the answer is the same.

We can now calculate the downstream spectrum. It is rather easier to work here with the integral spectrum $N(x,p)$, the number density of particles with momentum greater than p,

$$N(x,p) = \int_p^\infty 4\pi p'^2 f(x,p')\, dp'. \tag{2.48}$$

For simplicity (and with no real loss of generality) let us consider the case where all the particles are advected in from upstream with the same momentum p_0. Then upstream,

$$N(-\infty,p) = N_1(p) = \begin{cases} 0 & p>p_0 \\ N_0 & p<p_0 \end{cases} \tag{2.49}$$

and because particles are conserved and never lose energy, downstream

$$N(+\infty,p) = N_2(p) = \frac{U_1}{U_2}N_0 \qquad p<p_0. \tag{2.50}$$

Let p_n and v_n denote the momentum and associated velocity of a particle which returns from the downstream region n times and so makes a total of $2n$ crossings of the shock. In reality p_n is a random variable with a complicated distribution, but for large n and $v \gg U$ the distribution is sharply peaked and to first order in U/v we may regard it as determinate and given by

$$p_n \sim \prod_{i=1}^{n} [1+\tfrac{4}{3}(U_1-U_2)/v_i]p_0 \tag{2.51}$$

so that

$$\ln(p_n/p_0) \sim \tfrac{4}{3}(U_1-U_2)\sum_{i=1}^{n}\frac{1}{v_i}. \tag{2.52}$$

The probability of the particle reaching this momentum, i.e. of returning n times from downstream, is

$$P_n \sim \prod_{i=1}^{n}\left(1-\frac{4U_2}{v_i}\right) \tag{2.53}$$

so that

$$\ln P_n \sim -4U_2 \sum_{i=1}^{n} \frac{1}{v_i}$$

$$= -3 \frac{U_2}{U_1 - U_2} \ln (p_n/p_0) \qquad (2.54)$$

or

$$P_n = (p_n/p_0)^{-3U_2/(U_1-U_2)}. \qquad (2.55)$$

The number density of particles accelerated to momentum p_n or more is the total number density of particles, $N_2(p_0)$, multiplied by the probability of crossing the shock sufficiently often, P_n; thus

$$N_2(p_n) = P_n N_2(p_0) = \frac{U_1}{U_2} \left(\frac{p_n}{p_0}\right)^{-3U_2/(U_1-U_2)} N_0 \qquad p_n > p_0 \qquad (2.56)$$

and

$$f_2(p) = -\frac{1}{4\pi p^2} \frac{\partial N_2}{\partial p} = \frac{N_0}{4\pi} \frac{3U_1}{U_1 - U_2} \left(\frac{p}{p_0}\right)^{-3U_1/(U_1-U_2)} \qquad p > p_0 \qquad (2.57)$$

in agreement with the macroscopic derivation.

This derivation shows very clearly that, as in all Fermi acceleration processes, the key to obtaining a power law is that the momentum gained by a particle in each elementary acceleration event should be proportional to the momentum it already has and to its probability of escaping from the acceleration region. However, unlike other Fermi processes, the constant of proportionality, which determines the slope of the power law, is not arbitrary but is fixed by the kinematics of the shock.

3. Linear modifications

The elementary and idealised theory of the last section can be generalised and made more realistic in several ways. It is convenient to divide these extensions into two classes according to whether the reaction of the particles on the acceleration process is considered; this section deals with those in which the particles are treated as test particles and their reaction ignored. This assumption, that they move without influence on one another or on the background plasma, means that the problem of their transport in a given plasma system is linear and as usual this greatly simplifies the mathematical analysis. In the last section the shock was parallel, steady and plane and the particles moved very much faster than the background plasma, were scattered elastically, were not subject to additional energy loss processes and did not react on the plasma system; the modifications produced by relaxing all but the last of these conditions will now be examined.

3.1. Oblique shocks

The restriction to parallel shocks seems to be a major defect of the elementary theory (the average shock obliquity is some 60° if the magnetic-field direction is uncorrelated with the direction of shock propagation). Of course, it is clear that the acceleration mechanism continues to operate as long as particles can cross the front in both directions and be diffusively scattered on either side, but one might expect that the possibility of direct particle reflection from the shock would lead to significant alterations. Rather remarkably, at least to $O(U/v)$, this is not the case.

Consider an oblique shock where the point of intersection between a magnetic-field line and the front moves subluminally; then there exists a unique frame where the electric field vanishes, the fluid flows in upstream parallel to the magnetic field B_1 at velocity U_1 and angle θ_1 to the shock normal and exists downstream parallel to B_2 at velocity U_2 and angle θ_2 (figure 1(a)). In the macroscopic approach the transport equation (2.11) for this problem reduces to

$$\frac{\partial f}{\partial t}+U_i\cos\theta_i\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left((\kappa_\parallel\cos^2\theta_i+\kappa_\perp\sin^2\theta_i)\frac{\partial f}{\partial x}\right)\qquad x\neq 0 \tag{3.1}$$

where $i=1$ if $x<0$, 2 if $x>0$ and $\kappa_\parallel$, $\kappa_\perp$ are the diffusion coefficients parallel and perpendicular to the field (normally $\kappa_\parallel\gg\kappa_\perp$). The upstream and downstream solutions are clearly exactly the same as in the parallel case if normal components are used, i.e. if U is interpreted as $U\cos\theta$ and κ as $\kappa_\parallel\cos^2\theta+\kappa_\perp\sin^2\theta$. The problem of finding the correct matching conditions between these two solutions is not quite as easy as it is for the parallel shock because particles interact with an oblique shock in a non-trivial way. One can however argue that as long as the particles perceive the shock front merely as a kink in the magnetic field (unlike the low-energy particles constituting the background plasma which must be heated by collective effects in the front) the matching conditions must take the form of a linear relationship between the density f and the streaming S (in the shock frame) upstream and downstream:

$$\begin{pmatrix}f_+\\S_+\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}f_-\\S_-\end{pmatrix} \tag{3.2}$$

where $S=-\frac{1}{3}Up(\partial f/\partial p)-\kappa\,\partial f/\partial x$.

Liouville's theorem implies that a possible solution must be $f_+=f_-$, $S_+=S_-=0$, from which it follows that $a=1$, $c=0$. Requiring particle number conservation then shows that $d=1$. The remaining parameter, b, has the dimensions of an inverse velocity and is clearly of order $1/v$. Thus when, as in all the cases here considered, $S/f=O(U)$ this term is essentially of order U/v and can be neglected. One must however be slightly cautious; the distributions at the shock front cannot have the simple anisotropies presupposed in the diffusion approach so that, although the above argument shows that within the diffusion approach the only possible matching conditions are continuity of f and of the streaming, it is not obvious that this approximation can correctly describe acceleration at an oblique shock. Ignoring this worry, we see that within a macoscopic diffusion theory the upstream solutions, the downstream solutions and the matching conditions at the front for an oblique shock are all exactly the same as those for a parallel shock once normal components are used. Thus the spectrum is still determined by the same single parameter, the shock compression, and is essentially independent of the obliquity.

It is sometimes argued that if a fraction ϵ of particles incident on the front are reflected then the first matching condition should be

$$f(0+)=(1-\epsilon)f(0-) \tag{3.3}$$

(see Fisk 1971, Achterberg and Norman 1980) but this is clearly incorrect if the reflection is of the magnetostatic type considered here. The second boundary condition can probably be justified by measuring the streaming, not directly at the front, but a few mean free paths away (cf Toptyghin 1980); however, the argument is rather delicate and the more convincing proof of its correctness is that the same results follow if we repeat, as always working only to first order in U/v, the essential steps of the microscopic derivation.

Suppose a particle reaches the shock from upstream on a trajectory with pitch μ_1 and gyrophase α_1. It then interacts with the shock and is either transmitted with pitch and phase μ_2, α_2 or is reflected back with μ_3, α_3. If transmitted it diffuses in the downstream region until either escaping to ∞ or returning to the shock with μ_4, α_4 whereupon it is then transmitted back with μ_5, α_5 into the upstream region (figure 2). Denote by T_1 the set of μ, α values corresponding to upstream trajectories which lead to particle transmission through the shock, by R_1 those leading to reflection, by T_2 those downstream leading to transmission back upstream and let a superior bar denote the set of reversed trajectories; thus, for example,

$$\bar{T}_2=\{(\mu, \alpha)|(-\mu, \alpha)\epsilon T_2\}. \tag{3.4}$$

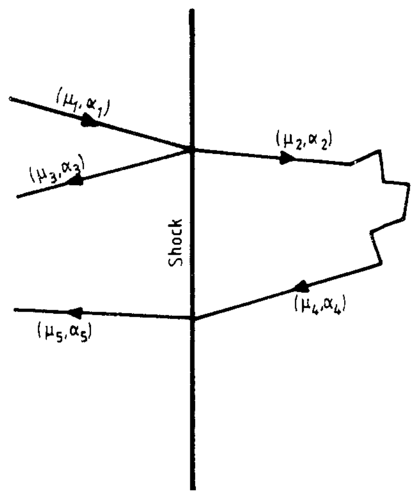

Figure 2. Schematic representation of the five sets of pitches and gyrophases used in the discussion.

Then clearly a particle which starts on a trajectory in T_1 after interacting with the shock proceeds downstream on a trajectory in $\bar{T}_2$ and one in R_1 gets reflected onto a trajectory in $\bar{R}_1$. Together T_1 and R_1 constitute the right hemisphere and T_2 is the left hemisphere (no reflection from downstream)

$$\begin{aligned} T_1 \cup R_1 &= \{(\mu, \alpha)|0<\mu \leqslant 1, 0 \leqslant \alpha < 2\pi\} \\ T_2 &= \{(\mu, \alpha)|-1 \leqslant \mu < 0, 0 \leqslant \alpha < 2\pi\}. \end{aligned} \tag{3.5}$$

Assuming that particles incident from upstream on the shock come from an almost isotropic and uniform distribution the probability of transmission is the ratio of the flux in T_1 to the total incident flux:

$$P_T = \int_{T_1} n\mu_1 v \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1/4\pi \left(\int_{T_1 \cup R_1} n\mu_1 v \, \mathrm{d}\mu_1 \wedge \mathrm{d}\alpha_1/4\pi \right)^{-1}. \tag{3.6}$$

By Liouville's theorem

$$\frac{1}{B_1}\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1=\frac{1}{B_2}\mu_2\,\mathrm{d}\mu_2\wedge\mathrm{d}\alpha_2. \tag{3.7}$$

Thus, transforming to the downstream variables,

$$P_T=\frac{1}{\pi}\frac{B_1}{B_2}\int_{\bar{T}_2}\mu_2\,\mathrm{d}\mu_2\wedge\mathrm{d}\alpha_2=\frac{B_1}{B_2}=\frac{\cos\theta_2}{\cos\theta_1} \tag{3.8}$$

and the flux into the downstream region is

$$\int_{T_1}\frac{nv}{4\pi}\cos\theta_1\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1=\frac{n}{4}v\cos\theta_2 \tag{3.9}$$

whereas the flux to ∞ is $nU_2\cos\theta_2$. Thus exactly as in the case of the parallel shock the probability of escape after transmission is $4U_2/v$. The combined probability that a particle incident from upstream will cross the shock and escape is therefore $4U_2\cos\theta_2/v\cos\theta_1$.

A particle which does not escape gains energy either by direct reflection, or by two transmissions across the shock. If its momentum with respect to the upstream fluid frame is p, then the expected momentum gain is

$$\begin{aligned}\pi v\langle\Delta p\rangle/p=&\int_{R_1}U_1(\mu_1-\mu_3)\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1+\int_{T_1}(\mu_1U_1-\mu_2U_2)\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1\\&+P_T\int_{T_2}(\mu_4U_2-\mu_5U_1)\mu_4\,\mathrm{d}\mu_4\wedge\mathrm{d}\alpha_4\end{aligned} \tag{3.10}$$

(the probability of escaping is $\mathrm{O}(U/v)$ and can be ignored in this calculation). But by repeated application of Liouville's theorem

$$\int_{R_1}-\mu_3\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1=\int_{\bar{R}_1}-\mu_3^2\,\mathrm{d}\mu_3\wedge\mathrm{d}\alpha_3=\int_{R_1}\mu^2\,\mathrm{d}\mu\wedge\mathrm{d}\alpha \tag{3.11}$$

and

$$\int_{T_1}-\mu_2\mu_1\,\mathrm{d}\mu_1\wedge\mathrm{d}\alpha_1=\frac{B_1}{B_2}\int_{T_2}-\mu_3^2\,\mathrm{d}\mu_2\wedge\mathrm{d}\alpha_2=-\frac{2\pi}{3}\frac{B_1}{B_2} \tag{3.12}$$

and

$$\int_{T_2}-\mu_5\mu_4\,\mathrm{d}\mu_4\wedge\mathrm{d}\alpha_4=\frac{B_2}{B_1}\int_{\bar{T}_1}-\mu_5^2\,\mathrm{d}\mu_5\wedge\mathrm{d}\alpha_5=\frac{B_2}{B_1}\int_{T_1}\mu^2\,\mathrm{d}\mu\wedge\mathrm{d}\alpha. \tag{3.13}$$

Thus

$$\frac{\langle\Delta p\rangle}{p}=\frac{4}{3}\left(\frac{U_1}{v}-\frac{U_2}{v}\frac{\cos\theta_2}{\cos\theta_1}\right). \tag{3.14}$$

Finally by exactly the same argument as in the case of the parallel shock the slope of the integral spectrum is given by the ratio of the escape probability to the mean momentum gain:

$$\frac{\mathrm{d}\ln N}{\mathrm{d}\ln p}=\frac{-4U_2\cos\theta_2/v\cos\theta_1}{\langle\Delta p\rangle/p}=\frac{-3U_2\cos\theta_2}{U_1\cos\theta_1-U_2\cos\theta_2} \tag{3.15}$$

which, when expressed in terms of the shock compression ratio, $r = U_1 \cos\theta_1/U_2 \cos\theta_2$, is independent of the obliquity. Thus if U is the conventional shock velocity (normal to the front, i.e. $U = U_1 \cos\theta_1$) the steady spectrum does not depend on the obliquity θ if $\sec\theta \ll v/U$ (this constraint can be very severe for almost perpendicular shocks).

3.2. *Time-dependent solutions*

A dimensional argument suggests that the basic time scale associated with diffusive shock acceleration should be of order κ/U^2 and this is confirmed by more detailed calculations (Forman and Morfill 1979, Krymsky *et al* 1979, Vasil'yev *et al* 1980, Axford 1981a, b). Following Axford (1981a, b) we study the model system of a steady planar shock in which a steady monoenergetic source of particles located in the shock is switched on at $t = 0$. Thus we look for time-dependent solutions of (2.31) with velocity profile (2.30), $f(t, x, p) = 0$ at $t = 0$ and a source $Q\delta(p - p_0)$ at $x = 0$.

On taking the Laplace transform with respect to time:

$$g(s, x, p) = \int_0^\infty \exp(-st) f(t, x, p)\,\mathrm{d}t \qquad (3.16)$$

(2.31) becomes upstream ($i = 1$) and downstream ($i = 2$)

$$sg + U_i \frac{\partial g}{\partial x} = \kappa_i \frac{\partial^2 g}{\partial x^2} \qquad (3.17)$$

(for simplicity we assume κ to be independent of x) of which the solutions satisfying the boundary conditions $g \to 0$ as $x \to \pm\infty$ are $g \propto \exp(\beta_i x)$ where

$$\beta_i = \frac{U_i}{2\kappa_i}\left[1 - (-1)^i \left(1 + \frac{4\kappa_i s}{U_i^2}\right)^{1/2}\right]. \qquad (3.18)$$

Let $g_0(s, p) = g(s, 0, p)$ be the transform of the spectrum at the shock; then the matching conditions

$$[f] = 0 \qquad \left[\kappa \frac{\partial f}{\partial x} + \frac{1}{3} Up \frac{\partial f}{\partial p}\right] = -Q\delta(p - p_0) \qquad (3.19)$$

imply that

$$\kappa_1 \beta_1 g_0 - \kappa_2 \beta_2 g_0 + \frac{1}{3}(U_1 - U_2) p \frac{\partial g_0}{\partial p} = \frac{1}{s} Q\delta(p - p_0). \qquad (3.20)$$

On letting $A_i = (1 + 4\kappa_i s/U_i^2)^{1/2} - 1$ this can be written

$$\frac{1}{2}(U_1 A_1 + U_2 A_2) g_0 + U_1 g_0 + \frac{1}{3}(U_1 - U_2) p \frac{\partial g_0}{\partial p} = \frac{1}{s} Q\delta(p - p_0) \qquad (3.21)$$

with solution

$$g_0(s, p_1) = \frac{3Q}{s(U_1 - U_2)}\left(\frac{p_1}{p_0}\right)^{-3U_1/(U_1 - U_2)} \exp\left(-\int_{p_0}^{p_1} \frac{3}{2} \frac{U_1 A_1 + U_2 A_2}{U_1 - U_2} \frac{\mathrm{d}p}{p}\right). \qquad (3.22)$$

The time-dependent spectrum of accelerated particles at the shock can now formally be obtained by inverting the transform:

$$f_0(t,p)=f(t,0,p)=\frac{1}{2\pi \mathrm{i}}\int_{-i\infty}^{+i\infty} g_0(s,p)\exp(ts)\,\mathrm{d}s \tag{3.23}$$

(the path of integration lies to the right of all the singularities of the integrand). As usual, the asymptotic behaviour at large times is obtained by looking only at the contribution from the rightmost singularity of the integrand, here the simple pole at $s=0$, which gives the steady spectrum

$$f_0(\infty,p)=\frac{3Q}{U_1-U_2}\left(\frac{p}{p_0}\right)^{-3U_1/(U_1-U_2)} \qquad p\geqslant p_0. \tag{3.24}$$

At the injection momentum p_0 this steady value is established immediately, i.e.

$$f_0(t,p_0)=f_0(\infty,p_0)=\frac{3Q}{U_1-U_2} \qquad t>0. \tag{3.25}$$

At a general time $t>0$ and momentum $p>p_0$ (3.22) shows that the spectrum has the form

$$f_0(t,p_1)=f_0(\infty,p_1)\int_0^t \varphi(t')\,\mathrm{d}t' \tag{3.26}$$

where

$$\varphi(t)=\frac{1}{2\pi \mathrm{i}}\int_{-i\infty}^{+i\infty}\exp[ts-h(s)]\,\mathrm{d}s \tag{3.27}$$

and

$$h(s)=\frac{3}{2}\int_{p_0}^{p_1}\frac{U_1A_1+U_2A_2}{U_1-U_2}\,\frac{\mathrm{d}p}{p}. \tag{3.28}$$

Physically we can interpret $\varphi(t,p_0,p_1)$ either as the probability distribution function for the time taken to accelerate a particle from momentum p_0 to p_1 or, by fixing t and regarding φ as a function of p_1, as the particle momentum spectrum (normalised to the steady spectrum) resulting from the acceleration over a time interval t of a single burst of particles injected at momentum p_0.

If we think of $\varphi(t)$ as the acceleration time distribution for given p_0, p_1, then from the obvious relations:

$$\int_0^\infty \varphi(t)\exp(-ts)\,\mathrm{d}t=\exp[-h(s)] \tag{3.29}$$

and $h(0)=0$, we see that

$$\int_0^\infty \varphi(t)\,\mathrm{d}t=1 \tag{3.30}$$

so that the distribution is correctly normalised. On differentiating with respect to s

and then setting $s=0$ we obtain an expression for the mean acceleration time:

$$\langle t\rangle = \int_0^\infty t\varphi(t)\,\mathrm{d}t = \frac{\partial}{\partial s}h(0)$$
$$= \frac{3}{U_1-U_2}\int_{p_0}^{p_1}\left(\frac{\kappa_1}{U_1}+\frac{\kappa_2}{U_2}\right)\frac{\mathrm{d}p}{p}. \tag{3.31}$$

Repeating this procedure we find the variance of the acceleration time:

$$\langle t^2\rangle - \langle t\rangle^2 = -\frac{\partial^2}{\partial s^2}h(0) = \frac{6}{U_1-U_2}\int_{p_0}^{p_1}\left(\frac{\kappa_1^2}{U_1^3}+\frac{\kappa_2^2}{U_2^3}\right)\frac{\mathrm{d}p}{p} \tag{3.32}$$

and in general the nth derivative of h at $s=0$ gives the nth cumulant of the probability distribution $\varphi(t)$.

It is interesting to note that if κ has a power law dependence on momentum, $\kappa \propto p^\alpha$, then for $p_1 \gg p_0$ the relative width of the distribution:

$$\frac{\langle (t-\langle t\rangle)^2\rangle}{\langle t\rangle^2} = \mathrm{O}(\alpha). \tag{3.33}$$

Thus, unless κ is almost independent of momentum ($\alpha \sim 0$) the peak in the acceleration time distribution is not very sharp.

For large values of s

$$h(s) = \beta\sqrt{s} - \frac{3}{2}\frac{U_1+U_2}{U_1-U_2}\ln\frac{p_1}{p_0} + \mathrm{O}(s^{-1/2}) \tag{3.34}$$

where

$$\beta = \frac{3}{U_1-U_2}\int_{p_0}^{p_1}(\sqrt{\kappa_1}+\sqrt{\kappa_2})\frac{\mathrm{d}p}{p} \tag{3.35}$$

and thus for small t

$$\varphi(t) \sim \left(\frac{p_1}{p_0}\right)^{3(U_1+U_2)/2(U_1-U_2)}\frac{\beta}{2\sqrt{\pi}}t^{-3/2}\exp(-\beta^2/4t)[1+\mathrm{O}(t)]. \tag{3.36}$$

The asymptotic behaviour of φ for large t is determined by the singularity of $h(s)$ with the most positive real part. Let

$$s_0 = \max_{p_0<p<p_1}\left(-\frac{U_1^2}{4\kappa_1}, -\frac{U_2^2}{4\kappa_2}\right) \tag{3.37}$$

then the relevant singularity is at s_0 and is a branch point of order $\nu = \frac{1}{2}$ or $\frac{3}{2}$. It follows that for large times

$$\varphi(t) \sim \frac{t^{-1-\nu}}{\Gamma(-\nu)}\exp[s_0 t - h(s_0)][1+\mathrm{O}(t^{-1/2})]. \tag{3.38}$$

These asymptotic forms for the wings of the distribution and the first few moments provide sufficient information for most purposes. An explicit expression for φ can be found in the special case κ independent of p and $A_1 = A_2$ (see Toptyghin (1980) and Axford (1981b) for details); the case $\kappa \propto p^\alpha$ can also be solved in terms of parabolic cylinder functions (M Forman, personal communication).

The most important conclusion to be drawn from expression (3.31) for the mean acceleration time is that the time scale for acceleration of particles of momentum p in an unmodified shock is

$$t_{\mathrm{acc}}(p)=\frac{3}{U_1-U_2}\left(\frac{\kappa_1}{U_1}+\frac{\kappa_2}{U_2}\right). \tag{3.39}$$

An interesting alternative derivation of this result is possible from the microscopic viewpoint (cf Lagage and Cesarsky 1981). Recall that the particle flux across the shock into the upstream region is $nv/4$. But the total number of particles in the upstream region (in a steady state) is $\int_{-\infty}^{0} n \exp(U_1x/\kappa_1)\,\mathrm{d}x = \kappa_1 n/U_1$. Therefore the mean residence time of each particle must be $(\kappa_1 n/U_1)\,(nv/4)^{-1} = 4\kappa_1/U_1 v$. This simple argument applied to the downstream region gives an infinite mean residence time, which is of course correct as there is a finite probability of escape. What we really require is the mean residence time of those particles which do not escape and to find this we need to know the probability of returning to the shock as a function of distance downstream.

Consider the steady solution of the diffusion equations corresponding to a source located a distance x_0 downstream and with an absorbing boundary at the origin:

$$\begin{gathered} U\frac{\partial n}{\partial x}=\kappa\frac{\partial^2 n}{\partial x^2}+Q\delta(x-x_0) \\ n(0)=0 \qquad n(\infty)<\infty. \end{gathered} \tag{3.40}$$

It is clearly

$$n=\begin{cases} \dfrac{Q}{U}[\exp(Ux/\kappa)-1]\exp(-Ux_0/\kappa) & 0\leqslant x\leqslant x_0 \\[2ex] \dfrac{Q}{u}[1-\exp(-Ux_0/\kappa)] & x_0\leqslant x<\infty \end{cases} \tag{3.41}$$

so that the flux to the origin is

$$\kappa\frac{\partial n}{\partial x}=Q\exp(-Ux_0/\kappa). \tag{3.42}$$

Thus for a particle inserted into the flow a distance x_0 downstream the probability of diffusing back to the origin must be $P_{\mathrm{ret}}=\exp(-Ux_0/\kappa)$. It follows that the number of particles downstream which will return to the shock is

$$\int_0^\infty P_{\mathrm{ret}}(x)n\,\mathrm{d}x=\kappa_2 n/U_2 \tag{3.43}$$

and thus the mean residence time downstream of these particles is $4\kappa_2/U_2 v$. We see that the mean time taken for a particle to complete one cycle of entering the downstream region, returning upstream and then re-entering the downstream region is

$$\Delta t=\frac{4}{v}\left(\frac{\kappa_1}{U_1}+\frac{\kappa_2}{U_2}\right). \tag{3.44}$$

With a mean momentum gain per cycle of

$$\Delta p = \frac{4}{3}\frac{U_1 - U_2}{v}p \tag{3.45}$$

this implies an acceleration time scale:

$$t_{\rm acc} = \frac{p\,\Delta t}{\Delta p} = \frac{3}{U_1 - U_2}\left(\frac{\kappa_1}{U_1} + \frac{\kappa_2}{U_2}\right). \tag{3.46}$$

3.3. *Non-planar shocks*

In the simple theory the shock front was taken to be planar; it is reasonably clear that the condition for this assumption to be valid is that the diffusion length scale, κ/U, be small compared to the radius of curvature of the front, R. In practice this tends to be the case if the shock is propagating and the time scales allow significant diffusive acceleration; the shock time scale is typically R/U and $R/U \gg \kappa/U^2$ implies $R \gg \kappa/U$. However, in view of the importance of spherical shocks, it is interesting to look at them in more detail. Two main cases have been considered: the blast wave from a point explosion and the shock terminating a stellar wind.

3.3.1. The spherical explosion. This is the more interesting case because of its possible application to the acceleration of cosmic rays in supernova blast waves. It has been studied by Krymsky and Petukhov (1980) and Prishchep and Ptsukin (1981), both of whom approximate the flow behind the blast wave by a linear velocity field. By further assuming the diffusion coefficient to be independent of momentum and position they obtain integral representations for the solution of the transport equation (in terms of confluent hypergeometric functions) which they then approximate for small values of the diffusion coefficient. It seems more direct to use the presence of a small parameter from the start of the analysis and this has the further advantage of allowing more general solutions to be studied.

The transport equation (2.11) in spherical coordinates is

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial r} = r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\kappa\frac{\partial f}{\partial r}\right) + \frac{1}{3}p\frac{\partial f}{\partial p}r^{1-d}\frac{\partial}{\partial r}(r^{d-1}U) \tag{3.47}$$

where κ is the radial component of the diffusion tensor, U is the radial velocity, d is the dimensionality ($d = 3$ is the only physically significant case, but it is as easy to do the calculations for general d and it is interesting to see how the solutions depend on the number of spatial dimensions) and spherical symmetry is assumed. We consider a self-similar velocity field of the form

$$U(r) = \begin{cases} V(\xi)\xi\dot{R} & 0 < \xi < 1 \\ 0 & \xi > 1 \end{cases} \tag{3.48}$$

where $\xi = r/R(t)$ is a dimensionless co-moving radial coordinate and $R(t)$ is the radius of the shock front at time t; (3.47) then becomes

$$\frac{R}{\dot{R}}\frac{\partial f}{\partial t} + (V-1)\xi\frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\epsilon\frac{\partial f}{\partial \xi}\right) + (d-1)\frac{\epsilon}{\xi}\frac{\partial f}{\partial \xi} + \frac{1}{3}\left(dV + \xi\frac{\partial V}{\partial \xi}\right)p\frac{\partial f}{\partial p} \tag{3.49}$$

where the dimensionless small parameter $\epsilon = \kappa/R\dot{R}$. The matching conditions at $\xi = 1$ are $[f] = 0$ and $[\epsilon\,\partial f/\partial \xi] = \frac{1}{3}V(1)p\,\partial f/\partial p$.

Let us suppose all particles enter the acceleration process below some momentum p_0; because the acceleration time scale is short we expect a power law spectrum to be established from p_0 up to some (time- and space-dependent) cut-off momentum. The amplitude of this spectrum is determined by the balance between advection out of the shock and advection and injection into the shock. If the seed particles are mainly advected in from a uniform external distribution we expect an almost constant spectrum, but if they are directly injected a dependence on the shock radius is quite possible. With these motivations (and because it works) we look for solutions with a time and momentum dependence of the form $f \propto p^{-a} R(t)^b$. Clearly in the limit $\epsilon \to 0$ a should approach the spectral slope found for the planar shock; the aim here is to find out how rapid this approach is, i.e. to calculate the first-order corrections to a.

With the above ansatz (3.49) becomes

$$\left[b + \tfrac{1}{3}a\left(dV + \xi\frac{\partial V}{\partial \xi}\right)\right]f + (V-1)\xi\frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi}\epsilon\frac{\partial f}{\partial \xi} + (d-1)\frac{\epsilon}{\xi}\frac{\partial f}{\partial \xi}. \tag{3.50}$$

This is a standard type of singular perturbation problem; direct expansion of f in powers of ϵ leads to 'outer' solutions which, in general, do not satisfy the full set of matching and boundary conditions and have to be matched to 'inner' boundary layer type expansions. The outer upstream solution is simply $f = 0$ (we are only looking at a momentum range where there are no upstream particles). The outer downstream solution is found by substituting $f = \Sigma_{i=0}^{\infty} f_i \epsilon^i$ in (3.50) and solving the resulting recurrence relations. The first equation

$$(1-v)\xi\frac{\partial f_0}{\partial \xi} = [b + \tfrac{1}{3}a(dV + \xi\partial V/\partial\xi)]f_0 \tag{3.51}$$

simply represents particle advection without diffusion. It thus has a trivial solution in Lagrangian coordinates which, expressed in terms of ξ, takes the more complicated form

$$f_0 = A\xi^{-ad/3}\exp\left((b + ad/3)\int_1^{\xi}\frac{d\xi'}{\xi'[1-V(\xi')]}\right)\left(\frac{1-V(\xi)}{1-V(1)}\right)^{-a/3}. \tag{3.52}$$

The higher terms, which we do not actually need, can in principle be found in terms of quadratures and represent small corrections for the limited diffusion which does take place. Clearly, a direct matching of the upstream and downstream outer solutions is only possible for the trivial solution, $f = 0$ everywhere, so that an inner solution is needed.

To obtain this expansion, valid in the neighbourhood of the shock front, we introduce a new local variable ζ by setting $\xi = 1 + \epsilon\zeta$. Then upstream, where $V = 0$, (3.50) becomes

$$\frac{\partial^2 f}{\partial \zeta^2} + \frac{\partial f}{\partial \zeta} = \epsilon\left(bf - \zeta\frac{\partial f}{\partial \zeta} - (d-1)\frac{1}{1+\epsilon\zeta}\frac{\partial f}{\partial \zeta}\right). \tag{3.53}$$

Substituting the expansion $f = \Sigma_{i=0}^{\infty}\epsilon^i f_i$ and imposing the boundary condition $f \to 0$ as $\zeta \to \infty$ (to match the outer upstream solution) and $f = A$ at $\zeta = 0$ (to match the outer downstream solution) we easily find the first two terms to be

$$\begin{aligned} f_0 &= A\exp(-\zeta) \\ f_1 &= -A\exp(-\zeta)[\tfrac{1}{2}\zeta^2 + (d+b)\zeta]. \end{aligned} \tag{3.54}$$

Thus to first order the inner upstream solution is

$$f = A \exp(-\zeta)\{1 - \epsilon[\tfrac{1}{2}\zeta^2 + (d+b)\zeta] + O(\epsilon^2)\} \qquad \zeta \geqslant 0. \tag{3.55}$$

This problem does not need an inner downstream expansion behind the shock (for the same reason that the downstream solution in the planar solution is constant), but for matching purposes it is convenient to re-express the outer solution in terms of the inner variable ζ. This gives

$$f = A\left[1 + \epsilon\zeta\left(b + \frac{ad}{3}V(1) + \frac{a}{3}V'(1)\right)[1 - V(1)] + O(\epsilon^2)\right]. \tag{3.56}$$

We now determine the spectral index, a, by joining the upstream and downstream solutions. The condition $[f] = 0$ has been automatically satisfied by choosing the same constant, A, in each region; the second condition, $[\partial f/\partial\zeta] = -aV(1)f/3$, gives

$$1 + \epsilon_+(d+b) + \epsilon_-\left(b + \frac{ad}{3}V(1) + \frac{a}{3}V'(1)\right)[1 - V(1)]^{-1} = \frac{a}{3}V(1) \tag{3.57}$$

or

$$a = \frac{3U_+}{U_+ - U_-}\left[1 + (d+b)\left(\frac{\kappa_+}{RU_+} + \frac{\kappa_-}{RU_-}\right) + \frac{V'(1)}{V(1)}\frac{\kappa_-}{RU_-}\right] + O(\epsilon^2) \tag{3.58}$$

where $+$ and $-$ denote values just in front of and just behind the shock, $U_+ = \dot{R}$ and $U_- = \dot{R}[1 - V(1)]$.

We see that in the limit $\epsilon \to 0$ the spectral slope is that of the planar shock, as expected. The first-order correction consists of two parts. The first, as can be seen from its proportionality to $d + b$ and the acceleration time scale t_{acc}, results from the finite rate at which particles gain energy; the accelerated particles were injected in the past when the shock was smaller and the injection rate different. The second, proportional to $V'(1)/V(1)$, results from adiabatic losses in the expanding flow behind the shock, but, at first sight rather surprisingly, only expansion above that due to the uniform linear enlargement of the shocked volume is significant (recall that $U = V\xi\dot{R}$). Prishchep and Ptuskin (1981) consider the case $b = 0$, $d = 3$, $V(\xi) = \text{constant}$ and obtain a first-order correction which agrees with the above.

It is worth noting that the Sedov solution for a strong point explosion in a gas with $\gamma = 5/3$ (Kahn 1975) gives $V(\xi) \sim 3(\xi^8 + 1)(3\xi^8 + 5)^{-1}$ so that $V'(1) = V(1) = 3/4$; thus for this important case

$$a = 4\left[1 + (3+b)\left(\frac{\kappa_+}{RU_+} + \frac{\kappa_-}{RU_-}\right) + \frac{\kappa_-}{RU_-}\right] + O(\epsilon^2) \tag{3.59}$$

and, with $a = 4$, the approximate downstream solution (3.52) is

$$f \sim A\xi^{6+5b/2} \exp[\tfrac{3}{16}(\xi^8 - 1)(b+4)]\left(\frac{8}{3\xi^8 + 5}\right)^{-4/3} p^{-4} \qquad p_0 < p < p_{\mathrm{cut\,off}}(\xi, t). \tag{3.60}$$

This shows that because of the downstream adiabatic deceleration the energetic particles are concentrated in a relatively thin shell behind the shock.

3.3.2. Stellar-wind terminal shocks. The other astrophysically significant case where acceleration at a spherical shock may be important is that of a stellar-wind terminal shock (Jokipii 1968, Fisk 1969, Cassé and Paul 1980, Webb *et al* 1982). A rough description of the dynamics of a stellar wind, sufficiently good for our purposes, is to say that after acceleration near the star the wind expands radially at constant velocity until its ram pressure drops to that of the external medium. A strong shock then separates the wind from the external flow which proceeds essentially at constant pressure (the flow downstream from a shock is subsonic) so that the velocity, if cooling is unimportant, goes approximately as the inverse square of the radius. Thus the radial velocity is

$$U=\begin{cases} U_1 & r<R \\ U_2(R/r)^2 & r>R \end{cases} \tag{3.61}$$

when R is the radius of the shock.

The transport equation downstream, i.e. outside the shock , is

$$U_2\left(\frac{R}{r}\right)^2\frac{\partial f}{\partial r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\kappa r^2\frac{\partial f}{\partial r}\right) \tag{3.62}$$

with the general solution

$$f=A(p)+B(p)\left[\exp\left(\int_R^r\frac{R^2U_2\,\mathrm{d}r'}{\kappa(r',p)r'^2}\right)-1\right]. \tag{3.63}$$

On setting

$$\Phi=\exp\left(\int_R^\infty\frac{R^2U_2}{\kappa r^2}\,\mathrm{d}r\right)-1$$

this gives

$$\begin{aligned} &f(R)=A \\ &f(\infty)=A+\Phi B \\ &\frac{\partial f}{\partial r}(R_+)=\frac{U_2}{\kappa}B. \end{aligned} \tag{3.64}$$

It should be noted that Φ, although it can be very large, is not infinite. Thus the spectrum at infinity must be specified as one of the boundary conditions and it is not enough merely to require that the solution be bounded downstream.

Inside the shock, $r<R$, there is strong adiabatic deceleration in the expanding wind and the transport equation is

$$U_1\frac{\partial f}{\partial r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa\frac{\partial f}{\partial r}\right)+\frac{2}{3}\frac{U_1}{r}p\frac{\partial f}{\partial p}. \tag{3.65}$$

The matching conditions at the shock are the usual ones, continuity of f and of the streaming. The inner boundary condition depends on the behaviour of κ. If $\kappa\to 0$ as $r\to 0$, then any particles near the origin suffer catastrophic adiabatic cooling and $f\to 0$ as $r\to 0$. However, if κ does not go to zero, or if the boundary condition is applied at a finite radius, it will be necessary to supply more information.

This problem is non-trivial. However, by assuming κ to be proportional to r and independent of momentum Webb *et al* (1982) were able to find a complete analytic solution. Although the answer is given in closed form and in terms of elementary functions it is extremely complicated and will not be reproduced here. Instead, as in the blast wave case, I will consider the 'small κ' limit as a perturbation to the plane shock theory.

Assume $\epsilon = \kappa_1(R)/U_1R$ is a small parameter and introduce the inner variable

$$\zeta = \frac{U_1}{\kappa_1(R, p)}(r-R). \tag{3.66}$$

Then to first order in ϵ the transport equation (3.65) becomes

$$\frac{\partial f}{\partial \zeta} = \frac{\partial^2 f}{\partial \zeta^2} + \epsilon\left(2\frac{\partial f}{\partial \zeta} + \beta\frac{\partial}{\partial \zeta}\zeta\frac{\partial f}{\partial \zeta} + \frac{2}{3}p\frac{\partial f}{\partial p} - \frac{2}{3}\alpha\zeta\frac{\partial f}{\partial \zeta}\right) \tag{3.67}$$

where

$$\frac{\partial \ln \kappa}{\partial \ln p} = \alpha \qquad \frac{\partial \ln \kappa}{\partial \ln r} = \beta. \tag{3.68}$$

The zeroth-order solution, as always, is $f_0 = A(p)\exp(\zeta)$ and the first-order solution is easily found to be

$$f = A\exp(\zeta)\left\{1 - \epsilon\left[\left(\frac{\beta}{2} - \frac{\alpha}{3}\right)\zeta^2 + \left(2 + \frac{2\alpha}{3} + \frac{2}{3}\frac{p}{A}\frac{\partial A}{\partial p}\right)\zeta\right]\right\}. \tag{3.69}$$

The inward streaming at the shock is

$$\begin{aligned}\kappa\frac{\partial f}{\partial r} + \frac{1}{3}p\frac{\partial f}{\partial p}U_1 &= U_1\left(\frac{\partial f}{\partial \zeta} + \frac{1}{3}p\frac{\partial f}{\partial \zeta}\right)\\ &= U_1A\left[1 - \epsilon\left(2 + \frac{2\alpha}{3} + \frac{2}{3}\frac{p}{A}\frac{\partial A}{\partial p}\right)\right] + \frac{1}{3}p\frac{\partial A}{\partial p}U_1\end{aligned} \tag{3.70}$$

and matching this to the streaming outside gives

$$\frac{1}{3}p\frac{\partial A}{\partial p}(U_1 - U_2 - 2\epsilon U_1) = -U_1A\left[1 - 2\epsilon\left(1 + \frac{\alpha}{3}\right)\right] + U_2B + Q \tag{3.71}$$

where Q is a source term representing injection at the shock and the U_2B term is a source due to inward diffusion from infinity. Combining the two sources, for injection at a single momentum p_0 we find an approximate power law spectrum above p_0 with slope

$$\frac{\partial \ln f}{\partial \ln p} = \frac{3U_1}{U_1 - U_2}\left[1 + 2\epsilon\left(\frac{U_2}{U_1 - U_2} - \frac{\alpha}{3}\right)\right]. \tag{3.72}$$

In this problem there also exists a tail of decelerated particles below p_0. To see this we have to introduce an 'inner' momentum variable as well. Let $p = p_0(1 + \frac{2}{3}\epsilon\Pi)$, then to zeroth order the transport equation reduces to

$$\frac{\partial f}{\partial \zeta} = \frac{\partial^2 f}{\partial \zeta^2} + \frac{\partial f}{\partial \Pi} \tag{3.73}$$

which is to be solved subject to the boundary conditions $f = \exp(\zeta)$ for $\zeta < 0$, $\Pi = 0$, continuity of the streaming across $\zeta = 0$ and $f \to 0$ as $\Pi \to -\infty$.

The solution is

$$f=\frac{1}{2}\left[\exp(\zeta)\operatorname{erfc}\left(\frac{\zeta+\Pi}{2\sqrt{\Pi}}\right)-1-\operatorname{erf}\left(\frac{\zeta-\Pi}{2\sqrt{\Pi}}\right)\right] \qquad \zeta<0, \Pi<0 \tag{3.74}$$

which shows the typical effects of modulation by the stellar wind.

These results agree with the exact solution found by Webb *et al* (1982) and, as can be seen from figure 3, reproduce its qualitative features well; above the injection energy the spectrum is a slightly steeper power law than the plane shock theory would predict and inside the shock there is a 'modulated' tail below the injection energy.

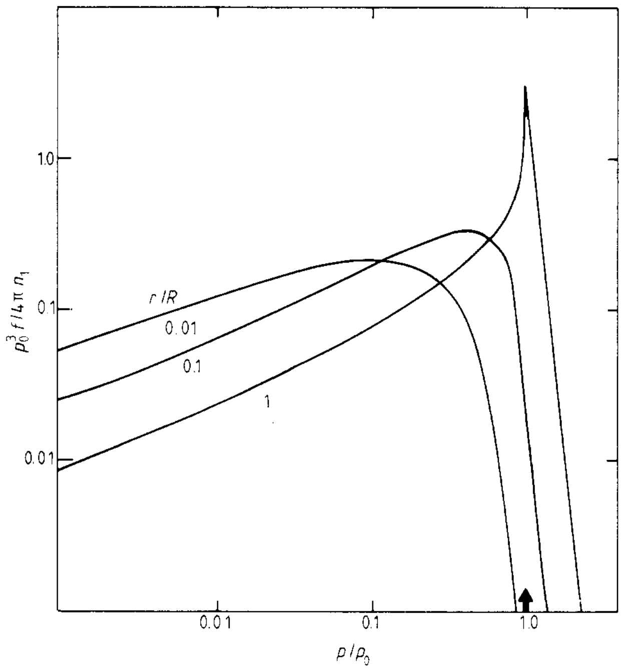

Figure 3. The spectrum of particles (at radius r) accelerated by a spherically symmetric stellar-wind terminal shock (radius R). The source of particles is a monoenergetic density n_1 at infinity, the shock has a compression ratio of four and the diffusion coefficient is assumed to equal Ur everywhere (from Webb *et al* 1982).

An interesting feature of this case is that the only reasonable source of 'seed' particles for acceleration is direct injection at the shock from the thermal background (cf § 4.4). Energetic particles released at the star, e.g. by flares, suffer strong adiabatic losses as they are advected to the shock and external cosmic rays have to diffuse inwards against the downstream flow which reduces their intensity by a factor $\Phi = O(\exp \epsilon^{-1})$.

3.4. Additional energy gains and losses

One can easily add terms to the basic transport equation (2.11) to represent other physical processes capable of changing a particle's momentum; obvious examples are synchrotron losses at high energy, ionisation losses at low energy, collisional losses and second-order Fermi acceleration in the downstream turbulence. However, it is not so easy to solve the resulting equations; simple exact solutions are unknown, but perturbation methods tend to work rather well. The crucial point is to compare the time scales. If the time scale associated with the added effect is longer than the diffusive acceleration time scale found in § 3.2, then the effect can be treated as a small perturbation in the same way that the adiabatic losses associated with spherical shocks were treated in § 3.3. If it is shorter, then it is the diffusive acceleration which must, both physically and mathematically, be treated as a perturbation. Because the two time scales usually have very different momentum dependencies the effect, in a first approximation, is simply to fix a cut-off momentum where the two are equal and above or below which the spectrum has the typical power law form of shock acceleration. This crude rule is probably good enough for most applications. Collisional losses, adiabatic cooling and synchrotron losses are discussed by Bulanov and Dogel' (1979); the effect of a collisional-type loss term, $-f/\tau$, has been investigated by Völk *et al* (1981); Webb (in preparation) has studied the effect of second-order Fermi acceleration and synchrotron losses.

3.5. *Effects of higher order in* (U/v)

As has been frequently emphasised the simple theory is correct only to $O(U/v)$. Actually the transport equation is correct to $O(U/v)^2$ (see, for example, Gleeson and Axford 1967) but the shock matching conditions for oblique shocks are known only to $O(U/v)$. It is desirable to have a theory correct to higher order, first, for discussions of injection (cf § 4.4) and, secondly, for applications involving relativistic shocks. Unfortunately, this is a very difficult problem as a glance at the pioneering paper of Peacock (1981) will show. An analytic theory can probably be developed for the physically rather uninteresting case of a non-relativistic parallel shock, but apart from this future developments will probably depend on numerical simulations (cf the work of Ellison (1981)).

4. Non-linear modifications

The previous section dealt extensively with modifications of the basic theory which did not consider the reaction of the accelerated particles. However, this cannot in general be ignored. It is easy to see from the elementary theory that the ratio of the downstream energetic particle pressure to that upstream is $a(a-3\gamma_c)^{-1}$ where $a = 3r(r-1)^{-1}$ is the slope of the power law and γ_c is the effective specific heat ratio of the particles (5/3 for non-relativistic, 4/3 for relativistic). Thus in the linear theory the downstream particle pressure diverges for strong shocks ($a \to 4$) and the reaction on the flow must be considered; even for moderate shocks the reaction must be considered if the upstream particle pressure is comparable to the gas pressure (as is observed in the interstellar medium). This means that the shock structure must be self-consistently calculated by including the cosmic-ray pressure in the fluid dynamics and cannot just be assumed given. Of course, the modification of the shock structure in turn implies changes in the spectrum of accelerated particles so that this is a highly non-linear problem.

There is a second type of reaction effect whch is not so non-linear—the effect of the energetic particles on the scattering. Streaming of particles through the background plasma leads to an instability which excites waves. These then produce enhanced scattering of the particles and, in principle, the particles can themselves generate the scattering centres needed for their acceleration.

This quasilinear problem is dealt with first and then the fully non-linear problems are considered. Throughout this section we consider only parallel one-dimensional shocks.

4.1. *Self-induced scattering*

The idea that the upstream scattering required for diffusive acceleration could be produced by self-excited Alfvén waves goes back to the paper of Bell (1978a). (There is no problem downstream, because behind a collisionless shock there is strong magnetic turbulence which produced plenty of scattering.) He considered the standard transport equation (note that the velocity of the scattering centres is $U - V$ where U is the fluid speed and V is the Alfvén speed; the self-excited waves all travel in the same direction as the streaming)

$$\frac{\partial f}{\partial t} + (U - V)\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\kappa \frac{\partial f}{\partial x}\right) \tag{4.1}$$

with an approximation from quasilinear theory for the diffusion coefficient

$$\kappa(p, x) = \frac{4}{3\pi} r_{\mathrm{g}} v \frac{B^2/8\pi}{E_{\mathrm{W}}} \tag{4.2}$$

where r_{g} is the particle gyroradius, v is its velocity and $E_{\mathrm{W}}(x, p)$ is the energy density of Alfvén waves (resonant with particles of momentum p) per unit logarithmic bandwidth. The wave intensity is given by

$$\frac{\partial E_{\mathrm{W}}}{\partial t} + (U - V)\frac{\partial E_{\mathrm{W}}}{\partial x} = (\sigma - \Gamma)E_{\mathrm{W}} \tag{4.3}$$

where

$$\sigma = \frac{4\pi}{3}\frac{V}{E_{\mathrm{W}}} p^4 v \frac{\partial f}{\partial x} \tag{4.4}$$

is an approximation from quasilinear theory for the growth rate due to the particle streaming and Γ represents damping of the waves.

Bell looked for a steady solution of these equations in which there was no damping of the waves and found the simple solution

$$\begin{aligned} f(x, p) &= f_1(p) + [f_2(p) - f_1(p)](1 - x/x_0)^{-1} \\ E_{\mathrm{W}}(x, p) &= \frac{V}{U - V}\frac{4\pi}{3} p^4 v [f_2(p) - f_1(p)](1 - x/x_0)^{-1} \end{aligned} \tag{4.5}$$

where

$$x_0 = \frac{1}{\pi^2}\frac{B^2/8\pi}{p^4 v [f_2(p) - f_1(p)]}\frac{v}{V} r_{\mathrm{g}}. \tag{4.6}$$

This shows what one would expect: the diffusion coefficient is small near the shock and increases (in fact, linearly) as one goes away from it. The quantity $(4\pi/3)p^4 v f(p)$ is the pressure per logarithmic bandwidth of particles with momentum p so that the energy density of the excited waves at the shock is exactly the cosmic-ray pressure increase divided by the Alfvén Mach number of the scattering centres. It is clear that the waves become non-linear for strong shocks or if the background magnetic field is weak (cf § 4.2.2.).

An interesting aspect of this solution, discovered by Lagage and Cesarsky (1982), is that the acceleration time scale associated with it is infinite. The reason is easy to see: the number of accelerated particles in the upstream region at some momentum

$$\int_{-\infty}^{0} \frac{n(0,p)\,\mathrm{d}x}{1-x/x_0}$$

diverges logarithmically. Thus the mean residence time upstream is infinite and the solution can never be established as a time-asymptotic state. Of course, in reality the diffusion coefficient cannot go on increasing indefinitely and some low level of pre-existing wave activity will provide a cut-off.

4.2. *Shock structure*

The harder problem, of determining a self-consistent shock structure, can also be traced back to one of the seminal papers on diffusive shock acceleration, that presented by W I Axford, E Leer and G Skadron at the 15th International Cosmic Ray Conference in Plovdiv (1977). This paper is, however, very short; more extensive discussions can be found in Drury and Völk (1981) and Axford *et al* (1982). Following these papers, we begin by looking at the simplest possible model in which this question can be asked.

4.2.1. The two-fluid model. The problem posed, namely how the reaction modifies the shock structure, is essentially one in fluid dynamics. This suggests that we try an approach in which the accelerated particles are treated as a fluid and the details of the spectrum ignored. The background plasma can then be described by the Euler equations with the reaction of the particles incorporated as an additional cosmic-ray pressure, P_C, in the momentum equation; thus

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho U)=0 \tag{4.7}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}(P_\mathrm{G}+P_\mathrm{C})=0 \tag{4.8}$$

$$\frac{\partial E_\mathrm{G}}{\partial t}+\frac{\partial}{\partial x}(UE_\mathrm{G})+P_\mathrm{G}\frac{\partial U}{\partial x}=0 \tag{4.9}$$

where ρ, U, P_G and E_G are the density, velocity, pressure and internal energy density of the background gas, respectively. The internal energy density is related to the pressure through an 'adiabatic exponent' or specific heat ratio γ_G:

$$P_\mathrm{G}=(\gamma_\mathrm{G}-1)E_\mathrm{G} \tag{4.10}$$

(for a monatomic non-relativistic gas with no internal degrees of freedom $\gamma_\mathrm{G}=5/3$).

The cosmic-ray pressure is given by one 'moment' of the distribution function:

$$P_\mathrm{C}=\frac{4\pi}{3}\int_0^\infty p^3 v f(p)\,\mathrm{d}p \tag{4.11}$$

and the internal energy density by

$$E_C = 4\pi \int_0^\infty p^2 T(p) f(p)\, dp \tag{4.12}$$

where T is the particle kinetic energy.

By taking this moment of the transport equation we obtain a 'hydrodynamic' equation for the cosmic rays:

$$\frac{\partial E_C}{\partial t} + \frac{\partial}{\partial x}(UE_C) + p_c \frac{\partial U}{\partial x} = \frac{\partial}{\partial x} \bar{\kappa} \frac{\partial E_C}{\partial x} \tag{4.13}$$

where

$$\bar{\kappa}(x) = \frac{\int_0^\infty \kappa(x, p) p^2 T\, (\partial f/\partial x)\, dp}{\int_0^\infty p^2 T\, (\partial f/\partial x)\, dp} \tag{4.14}$$

is an effective mean diffusion coefficient. Finally, the internal energy density is related to P_C through an 'adiabatic exponent' γ_C (4/3 for relativistic, 5/3 for non-relativistic particles) by

$$P_C = (\gamma_C - 1) E_C. \tag{4.15}$$

The propagation of small periodic disturbances in this system has been investigated by Ptuskin (1981). He shows that perturbations with very short wavelengths are practically decoupled from the cosmic rays and propagate at the gas sound speed, $(\gamma_G P_G/\rho)^{1/2}$, whereas very-long-wavelength perturbations couple to the cosmic rays and travel at the enhanced speed, $[(\gamma_G P_G + \gamma_C P_C)/\rho]^{1/2}$.

To investigate shock structures in this system we look for steady solutions representing transitions from one asymptotically uniform state (upstream) to another (downstream). Assuming that all time derivatives are zero we can at once write down four first integrals:

$$\rho U = A \tag{4.16}$$

$$AU + P_C + P_G = B \tag{4.17}$$

$$\frac{1}{2} AU^2 + \frac{\gamma_G}{\gamma_G - 1} UP_G + \frac{\gamma_C}{\gamma_C - 1} UP_C = C + \frac{\bar{\kappa}}{\gamma_C - 1} \frac{\partial P_C}{\partial x} \tag{4.18}$$

$$P_G U^{\gamma_G} = D. \tag{4.19}$$

The first three of these are simply statements of the conservation of mass, momentum and energy. The last, which states that the gas entropy is constant, is of a less fundamental nature and will, in general, only hold piece-wise; if a gas sub-shock has to be inserted in the transition, the entropy will increase there.

If we assume suitable values for the exponents γ_G and γ_C and the mean diffusion coefficient $\bar{\kappa}$ the above system consists of three algebraic equations and one differential equation in four unknowns. We can use the algebraic equations to eliminate in favour of one unknown (or combination of unknowns) and reduce the system to a single first-order ordinary differential equation in one unknown which then gives the shock structure (for details see Axford *et al* (1982)). An equivalent, but more geometrical, procedure is to use only the two linear algebraic equations to eliminate the density and the cosmic-ray pressure from the system and then to study the integral curves of the reduced system in the U, P_G plane. This has some advantages in discussing the insertion of gas sub-shocks.

In the U, P_G plane (see figure 4) the obvious inequalities $\rho \geq 0$, $P_C \geq 0$, $P_G \geq 0$ confine us to the triangular region bounded by the lines $U = 0$, $P_G = 0$ and $P_C = B - AU - P_G = 0$. The energy equation tells us that the diffusive component of the cosmic-ray energy flux is equal to a quadratic expression in U and P_G; thus it vanishes on a second-order curve in the diagram which is easily seen to be a hyperbola. The portions of this curve, which will be called the Hugoniot, lying within the triangle correspond to all possible uniform flows having the same mass, momentum and energy fluxes as the given one. The initial and final asymptotic states must lie on this curve, but this is the only information yielded by the conservation laws alone. Of course, if we specify that one of the pressures vanishes identically, then there are only two possible states (the intersections of the corresponding side of the triangle with the hyperbola) which define the normal Rankine–Hugoniot relations.

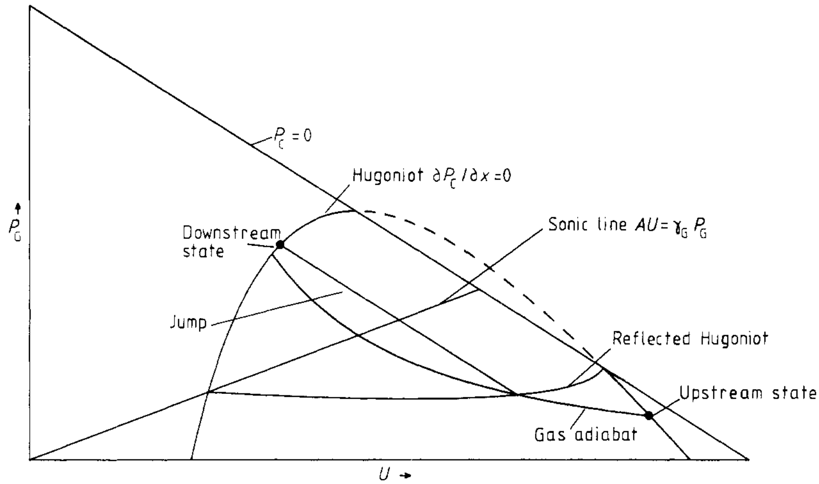

Figure 4. The shock construction described in the text; the figure has been drawn for the case $\gamma_C = 4/3$, $\gamma_G = 5/3$, $M = 2.0$ and $N = 0.3$ (from Drury and Völk 1981).

Thus to relate a given upstream state to a downstream state we must provide some physics beyond the basic conservation laws; in this simple model it consists of the assumption that the gas entropy only changes at gas sub-shocks and that the mean diffusion coefficient, $\bar{\kappa}$, is positive. In consequence, as the fluid flows through the transition from one uniform state to another, its state follows a path in the U, P_G diagram (note that this is a standard specific volume–pressure diagram) which consists of sections of gas adiabats connected by jumps corresponding to sub-shocks. The jump conditions at such a sub-shock are the ordinary Rankine–Hugoniot conditions for a pure gas shock, consistent with our assumption that there is no local energetically significant effect of a gas shock on the cosmic rays. Thus the cosmic-ray pressure and energy flux are continuous across the shock and the conditions are

$$[AU + P_G] = [P_C] = 0 \tag{4.20}$$

and

$$\left[\frac{1}{2}AU^2+\frac{\gamma_G}{\gamma_G-1}UP_G\right]=\left[\frac{\gamma_C}{\gamma_C-1}UP_C-\frac{\bar{\kappa}}{\gamma_C-1}\frac{\partial P_C}{\partial x}\right]=0. \tag{4.21}$$

If we temporarily use angle brackets for a mean as well as square brackets for a difference, i.e. $[U]=U_2-U_1$, $\langle U\rangle=(U_2+U_1)/2$, and note the identity

$$[xy]=[x]\langle y\rangle+\langle x\rangle[y] \tag{4.22}$$

the second condition can be written

$$A[U]\langle U\rangle+\frac{\gamma_G}{\gamma_G-1}([U]\langle P_G\rangle+\langle U\rangle[P_G])=0 \tag{4.23}$$

and, on using the first condition, this simplifies to

$$A\langle U\rangle=\gamma_G\langle P_G\rangle. \tag{4.24}$$

This means that in the U, P_G diagram the jump at a sub-shock must be parallel to the line $P_C=0$ and its mid-point must lie on the line $AU=\gamma_G P_G$. This line is the 'sonic line' where the local velocity equals the local gas sound speed, $(\gamma_G P_G/\rho)^{1/2}$, and the jump must be from the supersonic to the subsonic side.

The sonic line has, however, a second significance; on any given gas adiabat the cosmic-ray pressure has a single maximum which is attained where the adiabat intersects this line. This is easily seen by substituting the differential form of the adiabatic relation, $U\,dP_G+\gamma_G P_G\,dU=0$, into the differential form of the momentum conservation equation:

$$dP_C=-A\,dU-dP_G=-(AU-\gamma_G P_G)\,dU/U. \tag{4.25}$$

In the same way one can show that the quantity

$$\frac{1}{2}AU^2+\frac{\gamma_G}{\gamma_G-1}UP_G+\frac{\gamma_C}{\gamma_C-1}UP_C \tag{4.26}$$

has a single maximum on each gas adiabat when it intersects the line $AU=\gamma_G P_G+\gamma_C P_C$ (this is where the fluid velocity equals the propagation speed of a long-wavelength disturbance as found by Ptuskin (1981)). Thus, if an adiabat intersects the Hugoniot, it does so in two points, one on either side of this line.

We can now classify the possible types of transition (i.e. allowed shock structures) and obtain generalised Rankine–Hugoniot conditions. The key points are that as a fluid element goes through the transition its position coordinate, x, must increase monotonically, that equation (4.18) shows that $\partial P_C/\partial x$ is positive inside the Hugoniot curve and negative outside and that we know that on a gas adiabat P_C has a maximum at the sonic line.

The simplest possibility is that of a completely smooth transition. In the U, P_G diagram such a structure is represented by a section of a gas adiabat starting and ending on the Hugoniot. As the path lies inside the Hugoniot where $\partial P_C/\partial x$ is positive and as x increases monotonically the cosmic-ray pressure must also increase monotonically. It follows that such a solution can only exist if the section of adiabat does not cross the sonic line. This necessary condition is also sufficient because the differential equation can be inverted and integrated along such a path to yield a smooth shock structure. The simplest, but important and instructive, example is the cold plasma limit, $P_G=0$.

The section of gas adiabat then degenerates into part of the U axis and the integration is elementary. We have

$$AU+P_C=B \tag{4.27}$$

and

$$\frac{1}{2}AU^2+\frac{\gamma_C}{\gamma_C-1}UP_C=C+\frac{\bar{\kappa}}{\gamma_C-1}\frac{\partial P_C}{\partial x}. \tag{4.28}$$

Thus, on eliminating P_C in favour of U, $\partial U/\partial x$ is given by a quadratic in U:

$$\bar{\kappa}\frac{\partial U}{\partial x}=\frac{\gamma_C+1}{2}(U-U_1)(U-U_2) \tag{4.29}$$

where the zeros of the quadratic, U_1 and U_2, are the upstream and downstream speeds. An integration then shows that the shock structure is basically a hyperbolic tangent velocity profile:

$$U(x)=\frac{U_1+U_2}{2}-\frac{U_1-U_2}{2}\tanh\left(\frac{(1+\gamma_C)(U_1-U_2)}{4}\int_{x_0}^{x}\frac{dx'}{\bar{\kappa}(x')}\right). \tag{4.30}$$

The integration is elementary because the section of gas adiabat is straight; although this is exactly true only in the limit $P_G=0$, it is still a good approximation for weak shocks where the section of adiabat is so short that its curvature can be neglected. Thus we also obtain smooth shock structures of the same hyperbolic tangent type in the limit of infinitesimal shocks; with $\epsilon=(U_1-U_2)/(U_1+U_2)\ll 1$ the approximate solution is easily found to be (J F McKenzie, personal communication)

$$U(x)=U\left[1-\epsilon\tanh\left(\epsilon U\beta\int_{x_0}^{x}\frac{dx'}{\kappa(x')}\right)+O(\epsilon)^2\right] \tag{4.31}$$

with

$$\beta=\frac{\gamma_{C+1}}{2}+\frac{P_G}{P_C}\frac{\gamma_G(\gamma_G+1)}{2\gamma_C}. \tag{4.32}$$

The transition length scale ('the shock thickness') is inversely proportional to the shock strength and, for small cosmic-ray pressures, directly proportional to P_C. These infinitesimal shocks travel at the velocity found by Ptuskin for long-wavelength disturbances:

$$U=[(\gamma_G P_G+\gamma_C P_C)/\rho]^{1/2}. \tag{4.33}$$

Let us now consider the possibility of structures containing embedded gas sub-shocks. A sub-shock is represented in the diagram by a jump across the sonic line to the subsonic side. This jump must go directly to a point on the Hugoniot, i.e. the gas sub-shock must occur at the very end of the transition. To see this, suppose first that the subshock jumps to a point outside the Hugoniot. Then $\partial P_C/\partial x$ is negative and if x is to increase P_C must decrease. Thus the adiabatic section following the sub-shock leads away from the Hugoniot and cannot end before running into the region $P_C<0$. If, however, the sub-shock jumps to a point inside the Hugoniot, then P_C must increase and the adiabat inevitably runs into the sonic line. A second sub-shock is impossible because jumps must start on the supersonic side of the sonic line. The only possibility therefore is that the sub-shock jumps directly to the final downstream state.

In summary, the cosmic-ray reaction either smooths the shock structure completely, or the transition consists of a smooth part followed by an ordinary gas shock. The necessary and sufficient condition for the transition to be smooth is that the flow be everywhere supersonic. (This agrees with the general theory of Whitham (1974 p 353 *et seq*).) However, beyond this qualitative description, the above discussion tells us how to quantitatively relate upstream and downstream states; that is, it allows us to obtain generalised Rankine–Hugoniot conditions. Of course, they do not have quite the certainty that attaches to the gas dynamic Rankine–Hugoniot conditions (which are based only on the fundamental conservation laws) and they leave, as will be seen, certain questions open, but the value of such information is obvious.

In detail, the prescription for finding the possible downstream states compatible with a given upstream state is first to calculate the mass, momentum and energy fluxes A, B and C. With these values one then constructs the triangle of possible states, the sonic line and the Hugoniot curve. If the Hugoniot intersects the sonic line, one also constructs the locus of all points from which a sub-shock jump takes one directly to the Hugoniot (this curve is the sheared reflection in the sonic line of the Hugoniot, hence also a hyperbola; for brevity I will call it the reflected Hugoniot). Then one constructs the gas adiabat through the point representing the upstream state. All intersections of this adiabat with the reflected Hugoniot or the Hugoniot which lie on the supersonic side of the sonic line correspond to possible shock structures. One such intersection is, of course, the point representing the upstream state; this corresponds to the solution of no shock at all, i.e. the downstream state is the same as that upstream. Excluding this trivial solution it is clear that at least one other solution exists and that the number of non-trivial solutions must be odd (counting intersections with the appropriate multiplicity). It is, however, not obvious that the solution is unique, and indeed an analytic investigation of strong shocks in the limit of small cosmic-ray pressures (for details see Drury and Völk (1981)), or simply calculation, shows that in the case $\gamma_C = 4/3$, $\gamma_G = 5/3$ there can exist three possible solutions for certain upstream conditions. This lack of uniqueness should not be too surprising (the problem is, after all, non-linear).

The specification of the upstream state requires four dimensional parameters (ρ, U, P_G, P_C), but from these only two dimensionless combinations can be formed so that if we identify all upstream states which are related by pure scaling, the space of physically significant upstream states is two-dimensional and is conveniently parametrised by the shock Mach number M:

$$M^2 = \rho U^2 (\gamma_G P_G + \gamma_C P_C)^{-1} \tag{4.34}$$

(the ratio of the velocity of the shock with respect to the upstream fluid to the propagation speed of an infinitesimal shock) and the fractional contribution of the cosmic rays to the total pressure:

$$N = P_C (P_G + P_C)^{-1}. \tag{4.35}$$

The structure of the solution space, relative to this parametrisation, is sketched in figure 5. The vertical coordinate represents the downstream particle pressure (normalised to the total momentum flux) as a function of N and M over the physically significant region $0 \leqslant N \leqslant 1$, $M \geqslant 1$. This surface has a single cusp point where the gas adiabat and the reflected Hugoniot intersect with three-point contact and two fold

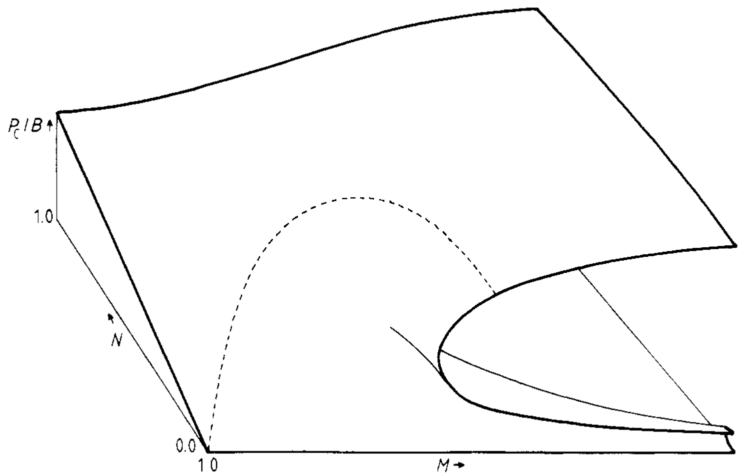

Figure 5. A sketch of the downstream ratio of the particle pressure to the total momentum flux as a function of the upstream parameters M and N. The dashed line on the surface indicates the approximate location of the boundary between those solutions with internal gas subshocks and those without.

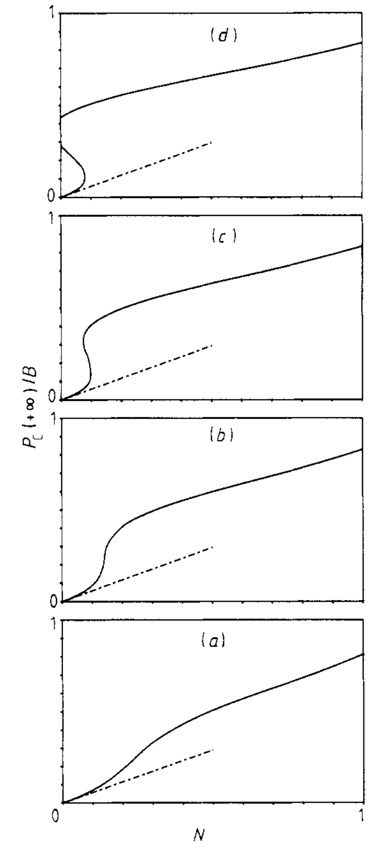

Figure 6. Calculated sections of the surface sketched in figure 5 at $M = 4.0(a)$, $5.0(b)$, $5.5(c)$ and $6.0(d)$. The chain line is an extrapolation from the linear test particle theory. This diagram shows clearly how non-linear effects enhance the acceleration and lead to multiple solutions in shocks with Mach numbers greater than about five (from Drury and Völk 1981).

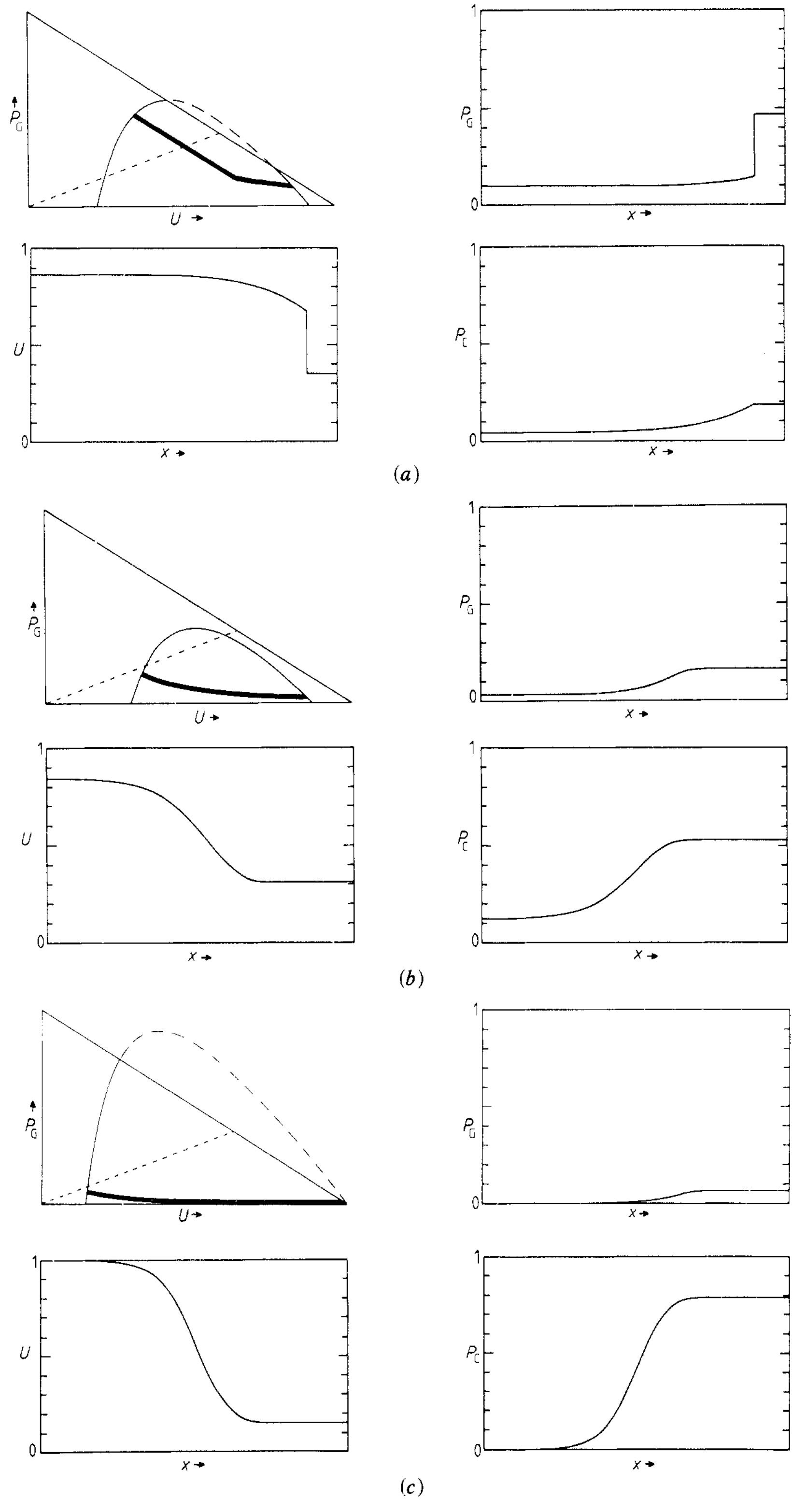

Figure 7. Three examples of shock structures: (a) $M = 2.0$, $N = 0.3$, (b) $M = 2.0$, $N = 0.8$, (c) $M = 13.0$, $N = 0.3$. The adiabatic exponent of the gas is 5/3, of the particles 4/3 and the solutions have been scaled to unit mass and momentum fluxes (from Drury and Völk 1981).

lines where they intersect with two-point contact. Between the two fold lines there exist three possible downstream states for every upstream state; at the fold lines two of these coalesce and disappear. The dashed line marks the boundary between smooth solutions and those which contain embedded sub-shocks. The results in figure 6 (which were calculated, not sketched) show sections of this surface at fixed M and figure 7 shows three typical shock structures.

How should we interpret these results physically? This is a hard question. It is important to realise that all we have done is to classify the *possible* downstream states, *if* there exist steady solutions and *if* the simple two-fluid model is applicable; the stability of these shock structures and the possibility of their creation have not been discussed. Indeed there are good reasons (see § 4.5) for supposing that if κ increases with p steady solutions cannot exist for strong shocks. Furthermore, the only thing we know about the cosmic rays in this simple two-fluid model is their pressure; their number density is unknown and this makes any discussion of, e.g. particle injection, very difficult. Nevertheless we can draw some tentative conclusions.

First, and most importantly, the model clearly indicates that shock waves tend to accelerate cosmic rays with high efficiency. The cosmic-ray diffusion is the major entropy generation process in both strong and weak shocks; only in moderate shocks (Mach 3 or so) are sub-shocks important. Taken literally, the model predicts that in very strong shocks 98% of the incoming kinetic energy is converted to cosmic-ray energy (with relativistic particles providing the dominant downstream pressure the shock has a compression ratio of 7 and the fraction of the incoming kinetic energy not available for conversion is $7^{-2} \sim 2\%$). While this almost certainly pushes the model beyond its limits of validity (unless κ is bounded) it shows that there is no obvious reason why the acceleration efficiency should be low.

A second important point is that the reaction modifies the acceleration in two ways (a point noted by Blandford (1979)). The smoothing of the shock decreases the acceleration (this is the effect one intuitively expects) but there is also a second, more subtle, effect which can increase the acceleration. If, as will normally be the case, the cosmic rays have a 'softer' equation of state than the gas ($\gamma_C < \gamma_G$) a cosmic-ray-dominated shock will have a larger compression ratio than a gas-dominated shock; the additional acceleration due to this increased compression can more than compensate for that lost due to the smoothing.

The conflict between these two effects provides a plausible physical explanation for the occurrence of multiple solutions in strong shocks with small upstream cosmic-ray pressure: either the structure is essentially that of a gas shock and the cosmic-ray pressure, though amplified, is small or the shock is cosmic-ray-dominated with the resulting increased acceleration itself producing this domination; between these two stable states there exists a third unstable state which, if perturbed, tends either to the gas-dominated or the cosmic-ray-dominated state. This accords with the fact that, if the upstream cosmic-ray pressure is too high, the only solution is the cosmic-ray-dominated one.

4.2.2. The two-fluid model with waves. The simple model analysed above assumes that the scattering of particles occurs from centres which move with the velocity of the fluid so that there is, in effect, a direct coupling between the cosmic rays and the gas. This is possible (for example, if the centres are tangential discontinuities), but we expect the scattering to be caused mainly by waves moving with respect to the gas. McKenzie and Völk (1981, 1982) and Völk and McKenzie (1981) have shown that, if these waves are Alfvén waves, they can be explicitly incorporated into a simple 'hydrodynamical' model which, as well as yielding a non-linear version of Bell's result for the wave generation, indicates how the approximation of neglecting the wave velocity could be justified.

As before we consider for simplicity a steady parallel shock and write down the three conservation laws:

$$\rho U = A \tag{4.36}$$

$$AU + P_G + P_C + P_W = B \tag{4.37}$$

$$\tfrac{1}{2}AU^2 + F_G + F_C + F_W = C. \tag{4.38}$$

The subscripts G, C and W refer to gas, cosmic rays and waves, the F are energy fluxes and the P are the pressures.

The gas energy flux consists of advection of the associated internal energy density E_G with the fluid velocity U plus a pressure work term:

$$F_G = UE_G + UP_G. \tag{4.39}$$

The cosmic-ray energy flux consists of the advection of internal energy at the velocity of the scattering centres, $U - V$, plus a pressure work term plus a diffusive flux:

$$F_C = (U - V)(E_C + P_C) - \bar{\kappa}\frac{\partial E_C}{\partial x}. \tag{4.40}$$

Finally the wave energy flux consists of the wave energy density advected at $(U - V)$ plus the pressure work term:

$$F_W = (U - V)E_W + UP_W. \tag{4.41}$$

The internal energies can be related to the pressures by

$$P_{C,G} = (\gamma_{C,G} - 1)E_{C,G} \tag{4.42}$$

and assuming the waves are Alfvén waves:

$$E_W = 2P_W \tag{4.43}$$

(see Dewar 1970; also Xiaoqing and Mutao 1982).

This gives a system of five unknowns (ρ, U and the three pressures) related by three conservation equations; to complete the system we need the equations describing the energy exchanges between the three components. These are

$$\frac{\partial}{\partial x}F_G = U\frac{\partial P_G}{\partial x} + L \tag{4.44}$$

the divergence of the gas energy flux is the rate at which the fluid does work against the gas pressure gradient plus the wave dissipation, L:

$$\frac{\partial F_C}{\partial x} = (U - V)\frac{\partial P_C}{\partial x}. \tag{4.45}$$

The divergence of the cosmic-ray energy flux is the rate at which the scattering centres do work against the cosmic-ray pressure gradient:

$$\frac{\partial F_W}{\partial x} = U\frac{\partial P_W}{\partial x} + V\frac{\partial P_C}{\partial x} - L. \tag{4.46}$$

The divergence of the wave energy flux is the rate at which the fluid does work against the wave pressure gradient plus the wave excitation by the cosmic-ray streaming minus the dissipation. Of course, these equations are not independent; if any two are assumed, the third can be deduced by using the differentiated forms of the conservation laws. The system is closed by prescribing some form for the dissipation, L, and noting that in a parallel shock the Alfvén speed, $V \propto U^{1/2}$. Clearly, the key feature which enables us to write down this system of equations is that Alfvén waves are non-dispersive; however, it seems plausible that, even if dispersive waves were important sources of scattering, these equations, with V interpreted as a 'mean wave speed', could be used as an approximate model.

The simplest case to consider is $L = 0$, i.e. no wave dissipation. The wave energy exchange equation can then be integrated to obtain a non-linear version of Bell's expression for the energy of the waves excited by the particle streaming. We have

$$\frac{\partial}{\partial x}[(3U - 2V)P_{\mathrm{W}}] = U\frac{\partial P_{\mathrm{W}}}{\partial x} + V\frac{\partial P_{\mathrm{C}}}{\partial x} \tag{4.47}$$

or

$$2(U - V)\frac{\partial P_{\mathrm{W}}}{\partial x} + P_{\mathrm{W}}\frac{\partial}{\partial x}(3U - 2V) = V\frac{\partial P_{\mathrm{C}}}{\partial x}. \tag{4.48}$$

Using the relation, $U \propto V^2$, we find that the integrating factor for the LHS is $(U - V)/V$ and thus

$$\frac{\partial}{\partial x}[2(U - V)^2 P_{\mathrm{W}}/V] = (U - V)\frac{\partial P_{\mathrm{C}}}{\partial x} = \frac{\partial}{\partial x}F_{\mathrm{C}}. \tag{4.49}$$

It follows that

$$\frac{(U - V)^2}{V}E_{\mathrm{W}} = F_{\mathrm{C}} + \text{constant} \tag{4.50}$$

(the energy equation can be used to eliminate F_{C} on the RHS; this gives the expression found by McKenzie and Völk). In agreement with the linear calculation of Bell, this shows that a fraction of the order of V/U of the energy given to the cosmic rays goes into exciting the waves.

This sounds relatively harmless and one is tempted to say that in the limit $V \ll U$ the wave excitation can be neglected and the simple two-fluid model used. This would, however, be premature: the problem is that the energy deposited in the waves, although small relative to the cosmic-ray energy, is much larger than that in the background magnetic field; the energy density available in the shock is of the order of $\frac{1}{2}\rho U^2$, if the acceleration is efficient this implies a wave energy density of the order of $\frac{1}{2}\rho UV$ whereas the background magnetic-field energy density is $\frac{1}{2}\rho V^2$. Clearly, if $V \ll U$, the waves become non-linear very near the start of the shock structure unless the shock is inefficient. What happens thereafter is an interesting but unanswered question†.

† I think it probable (but this is only speculation) that the magnetic field becomes so disordered that any attempt to decompose it into waves yields as many travelling forwards as backwards, the mean wave speed drops to zero and the irregularities stop growing. To put it another way, if the field is sufficiently disordered its irregularities are tied to the background plasma and cannot propagate through it like small-amplitude waves. However, it has been argued that just the opposite occurs, that the wave phase speed increases (Morrison *et al* 1981) due to 'trapping' effects (cf Völk and McKenzie 1981) and presumably diffusion is then an inappropriate model for the particle transport.

One possibility, which at least has the merit of allowing a relatively straightforward mathematical analysis, is to suppose that some efficient damping process transfers the excess energy from the waves to the gas.

4.2.3. Two-fluid model with strongly damped waves. If we assume that some, unspecified, damping process keeps the wave energy density (and hence also the wave pressure) small in comparison to the other energy densities the system of equations derived above can be simplified to

$$\begin{aligned} \rho U &= A \\ AU + P_C + P_G &= B \\ \tfrac{1}{2}AU^2 + F_C + F_G &= C \\ \frac{\partial}{\partial x} F_G &= U\frac{\partial P_G}{\partial x} + V\frac{\partial P_C}{\partial x} \\ \frac{\partial}{\partial x} F_C &= (U - V)\frac{\partial P_C}{\partial x}. \end{aligned} \tag{4.51}$$

This system can be analysed using a U, P_G diagram in exactly the same way as the simple two-fluid model and clearly in the limit $V \ll U$ the two theories become identical. The differences are quantitative rather than qualitative; the Hugoniot is not a simple hyperbola, but a curve which looks like a hyperbola (at least that part which lies inside the triangle of physical states) and instead of gas adiabats one has to use the integral curves of the energy transfer equation; the conclusions derived from the two-fluid model are essentially unchanged.

4.3. Non-linear effects on the particle spectrum

The simple hydrodynamic models discussed in the last subsection give some idea of how the shock structure reacts to the acceleration of energetic particles. The next problem is to see how this change in the structure influences the acceleration process and the spectrum of the accelerated particles. There are three obvious ways of tackling this problem; by looking for exact solutions, by trying asymptotic and perturbation expansions and by numerical methods. The first two of these have yielded useful information, the last has only begun to be applied, but will obviously be important in the future.

4.3.1. An exact solution. There is one special case of acceleration at a modified shock where a complete consistent solution is known (Drury *et al* 1982; see also Blandford and Payne 1981). The key point is that the steady-state transport equation

$$U\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\kappa\frac{\partial f}{\partial x} + \frac{1}{3}\frac{\partial U}{\partial x}p\frac{\partial f}{\partial p} \tag{4.52}$$

can be solved, in the sense that a useful expression can be found for the Green function relating the upstream and downstream spectra, if the velocity U and the diffusion coefficient κ are related by

$$\kappa\frac{\mathrm{d}U}{\mathrm{d}x} = \beta(U - U_1)(U - U_2) \tag{4.53}$$

where β is a constant. However, with $\beta = (\gamma_C + 1)/2$ this is exactly the shock structure equation of § 4.2 (the 'cold' plasma limit leading to smooth hyperbolic tangent type velocity transitions) and with β given by (4.32) the same equation applies in the weak shock limit. Thus in these cases a consistent analytic solution is possible if the diffusion coefficient is independent of momentum (so that $\bar{\kappa} = \kappa$).

To derive this solution we begin by changing the independent spatial variable from x to U so that (4.52) becomes

$$U\frac{\partial f}{\partial U} = \frac{\partial}{\partial U}\left(\kappa \frac{\mathrm{d}U}{\mathrm{d}x}\frac{\partial f}{\partial U}\right) + \frac{1}{3}p\frac{\partial f}{\partial p}. \tag{4.54}$$

The Euler operator $p\,\partial/\partial p$ suggests a Mellin transform:

$$g(\lambda) = \int_0^\infty p^{\lambda-1} f(p)\,\mathrm{d}p \tag{4.55}$$

and on using (4.53) we obtain the ordinary differential equation

$$U\frac{\partial g}{\partial U} = \beta\frac{\partial}{\partial U}\left((U-U_1)(U-U_2)\frac{\partial g}{\partial U}\right) - \frac{\lambda}{3}g \tag{4.56}$$

which has three regular singular points (at U_1, U_2, ∞) and can thus be reduced to Gauss's hypergeometric equation. The boundary conditions are applied at two of these singular points (g must be finite at U_2 and is known at U_1) and the solution is

$$g(U,\lambda) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)} F\left(a, b; c; \frac{U-U_2}{U_1-U_2}\right) g_1(\lambda) \tag{4.57}$$

where

$$\begin{aligned} 2a &= 1 - \frac{1}{\beta} - \left[\left(1-\frac{1}{\beta}\right)^2 + \frac{4\lambda}{3\beta}\right]^{1/2} \\ 2b &= 1 - \frac{1}{\beta} + \left[\left(1-\frac{1}{\beta}\right)^2 + \frac{4\lambda}{3\beta}\right]^{1/2} \\ c &= 1 + \frac{1}{\beta}\frac{U_2}{U_1-U_2} \end{aligned} \tag{4.58}$$

(for the theory of the hypergeometric equation see, for example, Whittaker and Watson (1902)). Thus

$$g_2(\lambda) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)} g_1(\lambda) \tag{4.59}$$

and it follows from the convolution theorem for the Mellin transform that

$$f_2(p) = \int_0^\infty f_1(p')G(p/p')\,\mathrm{d}p'/p \tag{4.60}$$

where $G(p/p')$, the Green function or spectrum produced by a monoenergetic upstream spectrum, is given by

$$G(p/p') = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\frac{p}{p'}\right)^{-\lambda}\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}\,\mathrm{d}\lambda. \tag{4.61}$$

In evaluating this integral it is convenient to set

$$
\begin{aligned}
s &= \ln(p/p') \\
\mu^2 &= \left(1-\frac{1}{\beta}\right)^2 + \frac{4\lambda}{3\beta} \\
\chi_{1,2} &= \frac{U_{1,2}}{U_1-U_2} \\
\chi_3 &= \chi_1+\chi_2 = \frac{U_1+U_2}{U_1-U_2}
\end{aligned}
\tag{4.62}
$$

the integration contour can then be deformed and the integral expressed as a sum of residues at the poles of one of the gamma functions with the result

$$
G = 3\chi_1 \frac{\exp[-3s\chi_1(1+\chi_2/\beta)]}{\Gamma(1+\chi_1/\beta)\Gamma(1+\chi_2/\beta)} \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(1+2n+\chi_3/\beta)\Gamma(1+n+\chi_3/\beta)\exp\{-3ns[\chi_3+\beta(n+1)]\} \tag{4.63}
$$

for $s>0$ and $G=0$ for $s \leqslant 0$. This rather complicated expression (which is, however, easy to compute as the terms in the series satisfy simple recurrence relations) shows that an upstream monoenergetic spectrum produces downstream a spectrum which is zero below the injection energy (no deceleration) and which consists of an infinite sum of power laws (in momentum) above this energy.

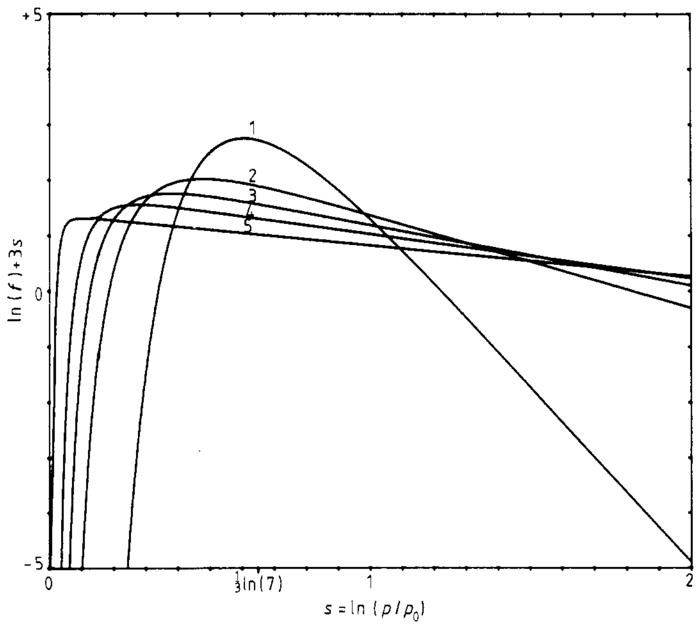

Figure 8. The downstream spectrum produced from an upstream delta distribution centred on p_0 by diffusive acceleration in a hyperbolic tangent type velocity transition of compression ratio 7. Solutions are shown for five values of the dimensionless diffusion coefficient β: (1) 0.1, (2) 0.5, (3) 1.0, (4) 2.0, (5) 10.0 (from Drury *et al* 1982).

Figure 8 shows the result of a numerical evaluation of this spectrum for the case where the overall compression ratio of the shock is seven and five values of β. This parameter can be regarded as a dimensionless diffusion coefficient; in the limit $\beta \to \infty$ the diffusion length scale is much larger than the shock thickness, the first term of the series dominates, the gamma functions disappear and

$$G \to 3\chi_1 \exp(-3s\chi_1) \tag{4.64}$$

which is, of course, the result found in the simple theory. But whereas that derivation assumed the shock to be thin compared to the particle gyroradius, this shows that in fact the result holds under the much weaker condition that the shock be thin compared to the diffusion length. In the other extreme, $\beta \to 0$, diffusion is relatively unimportant and the spectrum is the upstream delta distribution shifted by adiabatic compression a factor of the cube root of the compression ratio in momentum and broadened a little. However, in the consistent cases, β is of order unity (the diffusion length then determines the shock thickness). Perhaps the most interesting feature of the solutions is that, although the spectrum is not a simple power law, it can be very well approximated by one at momenta more than a few times the injection momentum; this is clear from figure 8 and is a consequence of the rapid convergence of the series (4.63) at large s. This asymptotic power law spectrum has a slope of

$$-3\chi_1(1+\chi_2/\beta) \tag{4.65}$$

i.e. steeper than in an unmodified shock of the same compression ratio.

It is also possible to derive an asymptote for the low momentum end of the spectrum and to show from the spectrum that particle number is conserved and that the pressure has the value required by the hydrodynamic equations; the details can be found in the original papers.

4.3.2. The high-energy asymptotic spectrum. We were able to find a solution of the transport equation in the last subsection only by assuming that the diffusion coefficient was momentum-independent. In reality κ will normally be an increasing function of p. For example, the energy dependence of the primary to secondary ratio in the galactic cosmic rays is usually taken as evidence that their diffusion coefficient has a power law dependence on momentum, $\kappa \propto p^{\alpha}$, with α in the range 0.3–0.5. In any case κ must eventually increase because it is bounded from below by the monotonically increasing value for Bohm diffusion ($\kappa \geqslant \kappa_B = \frac{1}{3}r_g v$). This suggests that if there is a steady shock structure with length scale L and velocities of the order of U, it might be useful to look for asymptotic expansions of the high-energy end of the spectrum by treating κ/LU as a large parameter.

For simplicity let us suppose $\kappa = \kappa_0(p/p_0)^{\alpha}$ where $\kappa_0/LU \gg 1$ and p_0 is so high that there are no pre-existing cosmic rays at momenta $p > p_0$. The problem is to study the asymptotic form of the solution of the steady transport equation (4.52) for $p > p_0$. For $|x| \gg L$, i.e. far upstream and downstream, $dU/dx = 0$ and the transport equation simplifies to the diffusion-advection equation. Thus upstream:

$$x \to -\infty \qquad f(x, p) \to f^-(p) \exp(U_1 x/\kappa) \tag{4.66}$$

and downstream:

$$x \to \infty \qquad f(x, p) \to f^+(p). \tag{4.67}$$

Let $g = \ln f$, $s = \ln(p/p_0)$; then (4.52) becomes

$$U\frac{\partial g}{\partial x} = \kappa\frac{\partial^2 g}{\partial x^2} + \kappa\left(\frac{\partial g}{\partial x}\right)^2 + \frac{1}{3}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\partial g}{\partial s}. \tag{4.68}$$

We now make the ansatz $g = \sum_{i=0}^{\infty} g_i\kappa^{-i}$ and obtain the recurrence relations

$$\frac{\partial^2 g_i}{\partial x^2} + \sum_{j=0}^{i}\frac{\partial g_j}{\partial x}\frac{\partial g_{i-j}}{\partial x} + \frac{1}{3}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\partial}{\partial s}g_{i-1} - \frac{(i-1)}{3}\frac{\mathrm{d}U}{\mathrm{d}x}g_{i-1} - U\frac{\partial}{\partial x}g_{i-1} = 0 \qquad i = 0, 1, 2\ldots. \tag{4.69}$$

The boundary conditions on the $g_i(x, s)$ follow from the large x behaviour of f:

$$\begin{aligned} &x \to +\infty \qquad g_i(x, s) \to g_i^+(s) \qquad i = 0, 1, 2, \ldots \\ &x \to -\infty \qquad g_i(x, s) \to g_i^-(s) \qquad i = 0, 2, 3, \ldots \\ &\qquad\qquad\quad\; g_1(x, s) \to g_1^-(s) + U_1 x. \end{aligned} \tag{4.70}$$

The first equation, $i = 0$, is

$$\frac{\partial^2 g_0}{\partial x^2} + \left(\frac{\partial g_0}{\partial x}\right)^2 = 0 \tag{4.71}$$

with solution

$$g_0(s, x) = g_0^-(s) = g_0^+(s). \tag{4.72}$$

The next equation is

$$\frac{\partial^2 g_1}{\partial x^2} + \frac{1}{3}\frac{\partial g_0}{\partial s}\frac{\mathrm{d}U}{\mathrm{d}x} = 0. \tag{4.73}$$

Integrating and using the boundary conditions

$$g_1 = g_1^+ - \frac{1}{3}\frac{\partial g_0}{\partial s}\int_{\infty}^{x}(U - U_2)\,\mathrm{d}x' \tag{4.74}$$

and if we choose the x origin such that

$$\int_{\infty}^{x}(U - U_2)\,\mathrm{d}x' \to (U_1 - U_2)x \qquad x \to -\infty \tag{4.75}$$

then

$$\frac{\partial g_0}{\partial s} = -3\frac{U_1}{U_1 - U_2}. \tag{4.76}$$

Thus

$$\begin{aligned} g_0 &= A - 3\chi_1 s \\ g_1 &= \chi_1\int_{\infty}^{x}(U - U_2)\,\mathrm{d}x' + g_1^+ \end{aligned} \tag{4.77}$$

using the notation of § 4.3.1.

Proceeding to $i = 2$

$$\frac{\partial^2 g_2}{\partial x^2} = \chi_1\chi_2(U_1 - U)(U - U_2) + \frac{1}{3}\frac{\mathrm{d}U}{\mathrm{d}x}\left(\alpha g_1 - \frac{\partial g_1}{\partial s}\right). \tag{4.78}$$

Integrating once and using the boundary conditions we obtain an equation for the s dependence of g_1:

$$\left(\frac{\partial}{\partial s}-\alpha\right) g_1^+ = -\chi_1(3\chi_2+\alpha)\Phi \tag{4.79}$$

where

$$\Phi = \int_{-\infty}^{\infty} \frac{(U_1-U)(U-U_2)}{U_1-U_2}\,\mathrm{d}x. \tag{4.80}$$

The solution is

$$\begin{aligned} g_1^+ &= \frac{\Phi}{\alpha}\chi_1(3\chi_2+\alpha) \qquad && \alpha \neq 0 \\ g_1^+ &= -3\chi_1\chi_2\Phi s && \alpha = 0 \end{aligned} \tag{4.81}$$

where we have dropped terms which only add a constant to g.

Thus the solution to first order is

$$g = A - 3\chi_1 s + \frac{1}{\kappa}\left(\chi_1 \int_{\infty}^{x} (U-U_2)\,\mathrm{d}x' + g_1^+\right) + \mathrm{O}(\kappa^{-2}) \tag{4.82}$$

or

$$f \propto \left(\frac{p}{p_0}\right)^{-3\chi_1} \exp\left(\frac{\chi_1}{\kappa}\int_{\infty}^{x} (U-U_2)\,\mathrm{d}x' + \frac{g_1^+}{\kappa} + \mathrm{O}(\kappa^{-2})\right). \tag{4.83}$$

It is interesting to note that this agrees with the exact solution found in § 4.3.1; if $\alpha = 0$ and $\kappa\, \mathrm{d}U/\mathrm{d}x = -\beta(U_1-U)(U-U_2)$ then $\Phi = \kappa/\beta$, $g_1^+ = -3\chi_1\chi_2\kappa s/\beta$ and the high-energy spectral slope is $-3\chi_1(1+\chi_2/\beta)$.

However, if κ increases with p, $\alpha > 0$, then the asymptotic slope is that predicted by the simple theory (as it clearly should be; but the point is of some importance as we will see). In principle, the series could be continued to higher orders, but the algebra grows at an alarming rate and there is no reason to suppose that the result would be very useful.

4.3.3. Blandford's perturbation solution. A different approach has been taken by Blandford (1980) who regards the cosmic-ray reaction as a small perturbation and simultaneously expands the shock structure and the spectrum in powers of the ratio of the downstream cosmic-ray pressure to the kinetic momentum flux. This has the obvious disadvantage of only working when the shock is an inefficient accelerator of cosmic rays (a circumstance which the considerations of § 4.2 suggest is rarely the case) and leads to rather complicated algebra. However, it does yield more information in those cases where it is applicable than the analyses of §§ 4.2 and 4.3.2.

We begin by supposing that in terms of a small parameter ϵ we can make the expansions:

$$\begin{aligned} U &= U^{(0)} + \epsilon U^{(1)} + \epsilon^2 U^{(2)} \ldots \\ f &= \epsilon f^{(0)} + \epsilon^2 f^{(1)} + \ldots \\ P_\mathrm{G} &= P_\mathrm{G}^{(0)} + \epsilon P_\mathrm{G}^{(1)} + \ldots \\ P_\mathrm{C} &= \int_0^{\infty} \frac{4\pi}{3} p^3 v f\,\mathrm{d}p = 0 + \epsilon P_\mathrm{C}^{(1)} + \ldots . \end{aligned} \tag{4.84}$$

On inserting these into the transport and dynamic equations to zeroth order we obtain the simple theory of acceleration at a gas shock with no reaction. Thus from § 2.3.1 the solution is (for simplicity we assume κ independent of x)

$$
\begin{aligned}
U^{(0)}(x) &= \begin{cases} U_1^{(0)} & x<0 \\ U_2^{(0)} & x>0 \end{cases} \\
P_G^{(0)}(x) &= \begin{cases} P_{G1}^{(0)} & x<0 \\ P_{G2}^{(0)} & x>0 \end{cases} \\
f^{(0)}(x,p) &\propto \begin{cases} \left(\dfrac{p}{p_0}\right)^{-3\chi_1} \exp(U_1 x/\kappa) & x<0 \\ \left(\dfrac{p}{p_0}\right)^{-3\chi_1} & x>0 \end{cases}
\end{aligned}
\tag{4.85}
$$

where $\chi_1 = U_1/(U_1-U_2)$ and $U_{1,2}^{(0)}$, $P_{G1,2}^{(0)}$ are related by the Rankine–Hugoniot conditions.

To obtain the first-order corrections to the shock structure it is convenient to use the integrated forms of the mass and momentum conservation laws:

$$
\begin{aligned}
\rho U &= A \\
AU + P_G + P_C &= B.
\end{aligned}
\tag{4.86}
$$

The two scaling degrees of freedom can be used to hold A and B constant (so that they do not need to be expanded) and, following Blandford, we will also hold the shock (gas) Mach number constant (thus $U_1 = U_1^{(0)}$, $P_{G1} = P_{G1}^{(0)}$). By using the adiabatic relation $P_G \propto U^{-\gamma}$ it is then easy to show that upstream

$$
\begin{aligned}
U^{(1)} &= -U_1^{(0)}\theta(x) \\
P_G^{(1)} &= \gamma P_{G1}^{(0)}\theta(x) \\
P_C^{(1)} &= (AU_1^{(0)} - \gamma P_{G1}^{(0)})\theta(x)
\end{aligned}
\tag{4.87}
$$

where we define ϵ by imposing the condition $\theta(0)=1$. Thus

$$
\theta(x) = \frac{\int_0^\infty p^{-3\chi_1} \exp(U_1 x/\kappa) p^3 v \, \mathrm{d}p}{\int_0^\infty p^{-3\chi_1} p^3 v \, \mathrm{d}p}.
\tag{4.88}
$$

Applying the Rankine–Hugoniot conditions to the sub-shock we then obtain the (x-independent) downstream perturbations

$$
\begin{aligned}
U_2^{(1)} &= -U_2^{(0)} + \frac{2\gamma}{A} P_{G1}^{(0)} \\
P_{G2}^{(1)} &= \gamma P_{G2}^{(0)} - 2AU_1^{(0)} \\
P_{C2}^{(1)} &= AU_1^{(0)} - \gamma P_{G1}^{(0)}
\end{aligned}
\tag{4.89}
$$

(note that $AU_2^{(1)} + P_{G2}^{(1)} + P_{C2}^{(1)} = 0$ as required by the constancy of B).

The first-order terms in the transport equations give

$$
\begin{aligned}
&U^{(0)}\frac{\partial f^{(1)}}{\partial x} - \kappa\frac{\partial^2 f^{(1)}}{\partial x^2} - \frac{1}{3}\frac{\partial U^{(0)}}{\partial x} p \frac{\partial f^{(1)}}{\partial p} \\
&\quad = -U^{(1)}\frac{\partial f^{(0)}}{\partial x} + \frac{1}{3}\frac{\partial U^{(1)}}{\partial x} p \frac{\partial f^{(0)}}{\partial p} = \psi(x,p)
\end{aligned}
\tag{4.90}
$$

i.e. the spectral correction is obtained by solving the zeroth-order transport equation with a source term representing the non-linear effects. The results of § 2.3.1 allow us to write down the Green function for the zeroth-order transport equation with an upstream source, $\delta(x-x')\delta(p-p')$ with $x'\leqslant 0$:

$$\begin{aligned}G(x,p,x',p') &= \frac{1}{U_1}[1-\exp(U_1x/\kappa)]\\ &\quad\times\{\exp(U_1(x-x')/\kappa]H(x'-x)+H(x-x')\}H(-x)\delta(p-p')\\ &\quad+[\exp(U_1x/\kappa)H(-x)+H(x)]\frac{3\chi_1}{U_1}\left(\frac{p}{p'}\right)^{-3\chi_1}H(p-p')\frac{1}{p'}\end{aligned} \tag{4.91}$$

in terms of which the first-order correction can be expressed purely in terms of quadratures:

$$f^{(1)}(x,p)=\int_{-\infty}^{0}\mathrm{d}x'\int_0^p \mathrm{d}p' G(x,p,x',p')\psi(x',p'). \tag{4.92}$$

The simplest case to consider is that where κ is independent of p. Then $\theta(x)=\exp(U_1^{(0)}x/\kappa)$ and the source term

$$\psi(x,p)=f^{(0)}(x,p)\theta(x)\left(\frac{U_1^{(0)2}}{\kappa}(1+\chi_1)-(U_1^{(0)}-U_2^{(0)}+2TU_1^{(0)})\chi_1\delta(x)\right). \tag{4.93}$$

The parameter $T=\gamma P_{\mathrm{G}1}^{(0)}/AU_1^{(0)}$ is, as in Blandford's paper, the inverse square Mach number of the shock. Thus, the downstream perturbation to the spectrum:

$$f^{(1)}(0,p)=f^{(0)}(0,p)3\chi_1(\chi_2/2-2T\chi_1)\ln(p/p_0) \tag{4.94}$$

and to first order

$$\begin{aligned}f(0,p)&=f^{(0)}+\epsilon f^{(1)}\\ &\propto\left(\frac{p}{p_0}\right)^{-3\chi_1[1-\epsilon(\chi_2/2-2T\chi_1)]}.\end{aligned} \tag{4.95}$$

We see that for weak shocks the spectrum steepens but for strong shocks it flattens. The above expression agrees with that found by Blandford if a misprint in his equation (21) is corrected.

The case $\kappa\propto p^{\alpha}$ can also be considered, but the integrals involved have to be performed numerically. However, one can easily see that if κ is an increasing function of p the downstream spectral slope at high momenta will be that appropriate to the overall compression ratio of the transition:

$$\frac{\partial\ln f}{\partial\ln p}\to-\frac{3U_1^{(0)}}{U_1^{(0)}-U_2^{(0)}-\epsilon U_2^{(1)}}=-3\chi_1[1-\epsilon(\chi_2-2T\chi_1)] \tag{4.96}$$

whereas at low momenta it tends to that appropriate to the compression ratio of the sub-shock:

$$\frac{\partial\ln f}{\partial\ln p}\to-\frac{3U_1^{(0)}(1-\epsilon)}{U_1^{(0)}(1-\epsilon)-U_2^{(0)}+\epsilon U_2^{(0)}-2T\epsilon U_1^{(0)}}=-3\chi_1(1+2\epsilon T\chi_1). \tag{4.97}$$

4.4. *Injection and selective acceleration*

It seems very probable, both on theoretical grounds (the collective processes which heat the plasma within a collisionless shock do not necessarily produce Maxwellian distributions) and by extrapolation from observations of the Earth's bow shock, that if the shock structure contains a collisionless sub-shock this injects particles from the background plasma into the diffusive acceleration mechanism. However, it is very hard to quantify this effect, although interesting attempts have been made by Eichler (1979), Bulanov and Dogel' (1979) and Krymsky (1981) analytically and by Ellison (1981) numerically: while all agree that injection from the thermal background allows a strong shock, even with no upstream 'seed' cosmic rays, to accelerate cosmic rays 'efficiently' this result must largely be attributed to their common assumption that the process which scatters the high-energy particles is the same as that which heats the background plasma (whereas very different interactions could be involved). Until we have some theoretical understanding of the microscopic structure of a quasi-parallel collisionless shock we can probably do no better than follow Bell (1978b) and assume that some small fraction (say, 10^{-3}) of the thermal particles advected into the shock obtain enough energy to enter the diffusive mechanism (cf Sonnerup 1979). From various observational constraints the existence of such a process is probably essential if diffusive shock acceleration is to be regarded as a viable theory for the origin of the galactic cosmic rays, a point emphasised by Eichler (1980).

An interesting aspect (Eichler 1979, Ellison 1981), assuming injection to exist, is the question of selectivity in the acceleration: how is the composition of the accelerated particle at high energies related to that of the background plasma? There are clearly two effects here; the initial injection could be selective and if the shock structure has been modified the acceleration from the injection energy up to the observation energy will select those species with larger diffusion coefficients. Both effects suggest that the abundance of a species in the high-energy particles relative to that in the plasma should be a smooth, and probably increasing, function of its mass to charge ratio, but as with the injection itself there exists no quantitative theory.

The phenomenological approach to injection suggested above can, of course, be incorporated into the two-fluid model of the shock structure discussed in § 4.2 (H J Völk, personal communication). One simply says that a fraction $\theta (0 \leqslant \theta < 1)$ of the kinetic energy dissipated in the sub-shock goes into cosmic-ray energy so that the sub-shock jump conditions become

$$[P_C] = 0 \qquad [F_C] = -\theta[\tfrac{1}{2}AU^2] \tag{4.98}$$

or, in terms of the gas,

$$\begin{gathered} A[U] + [P_G] = 0 \\ \tfrac{1}{2}A[U^2] + (1-\theta)[F_G] = 0 \\ \Rightarrow A(1-\theta+\theta\gamma_G)\langle U\rangle = \gamma_G\langle P_G\rangle. \end{gathered} \tag{4.99}$$

Thus the jump in the U, P_G diagram is still parallel to $P_C = 0$, but its midpoint now lies on the new line (assuming θ to be a constant):

$$A[1+\theta(\gamma_G - 1)]U = \gamma_G P_G. \tag{4.100}$$

With this new line one can then construct a new reflected Hugoniot and carry out the construction of § 4.2. However, whereas in § 4.2 the existence of at least one non-trivial solution was guaranteed, with $\theta > 0$ there exist some upstream states with no non-trivial solutions. The problem is that the cosmic-ray pressure still reaches its maximum on each gas adiabat at the old sonic line whereas weak subshocks, because of the energy drain due to injection, travel at less than the sound speed. The effect probably does not have much physical significance, but it serves as a warning not to assume that steady solutions must exist, a point examined in the next subsection.

Eichler (1979) claims, on the basis of his simplified model, that the reaction regulates the injection so that the overall acceleration efficiency is kept at about 50%. This is a very appealing idea, but unfortunately reaction effects are more complicated than his model allows (see the next subsection) so that without further work his conclusions cannot be accepted.

4.5. *Pressure divergence in non-linear shocks*

In § 4.2 we discussed how the reaction of the accelerated particles, expressed in terms of a non-zero cosmic-ray pressure, could modify steady shock structures. Then in § 4.3 we examined particle acceleration in these modified steady shocks. We must now check whether this is consistent and the spectral modifications are compatible with the assumed cosmic-ray pressures. When we do this we find that we are led to a contradiction if κ increases with p and the shock is strong. The problem is simple: if we assume a strong shock accelerating cosmic rays to have a steady structure, then because of the cosmic rays' softer equation of state the overall compression ratio of the transition will exceed four; but if κ increases with p then the high-energy asymptotic slope of the spectrum is determined by the overall compression ratio. It will thus be flatter than $f(p) \propto p^{-4}$ and the pressure integral

$$P_C = \frac{4\pi}{3} \int_0^\infty p^3 v f \, dp$$

will diverge. We are forced to conclude that no steady structure can exist for strong shocks accelerating cosmic rays unless κ is bounded from above (or additional effects are incorporated, e.g. a loss mechanism at high energies, which can provide a suitable cut-off).

It is interesting to speculate about what sort of unsteady structure such a shock might have; the obvious alternatives are that the structure is periodic or that it changes secularly. I know of no detailed solutions relevant to this problem, but some information can be gleaned from the time dependence calculation of § 3.2.

Let us suppose that the diffusion coefficient $\kappa \propto p^\alpha$ and consider a burst of particles with momentum p_0 released at $t = 0$ into an unmodified shock. The particle spectrum at the shock is then

$$f \propto \left(\frac{p}{p_0}\right)^{-3x_1} \varphi(t, p, p_0) \tag{4.101}$$

in the notation of § 3.2. But if $\alpha > 0$, then at large times the system 'forgets' the injection momentum and the acceleration time distribution tends to a self-similar form:

$$\varphi(t, p, p_0) \to t_a^{-1} \tilde{\varphi}(t/t_a) \tag{4.102}$$

where the acceleration time $t_a \propto \kappa \propto p^\alpha$. The cosmic-ray pressure at the shock:

$$P_C \propto \int_0^\infty p^{3(\gamma_C - x_1)} \varphi(t, p, p_0) \, dp/p \tag{4.103}$$

where $\gamma_C = 4/3$ if the particles are relativistic, $5/3$ if they are non-relativistic.

Thus, as $t \to \infty$,

$$P_C \propto \int_0^\infty p^{3(\gamma_C - \chi_1)} t_a^{-1} \tilde{\varphi}(t/t_a) \frac{dp}{p}$$

$$\propto \int_0^\infty (t/\tau)^{3(\gamma_C - \chi_1)/\alpha - 1} \tilde{\varphi}(\tau) \frac{d\tau}{\tau}$$

$$\propto t^{3(\gamma_C - \chi_1)/\alpha - 1}. \tag{4.104}$$

An alternative microscopic derivation of this power law time dependence is perhaps more convincing. The particles gain momentum at a rate

$$\frac{1}{p}\frac{dp}{dt} = \frac{U_1 - U_2}{3}\left(\frac{\kappa_1}{U_1} + \frac{\kappa_2}{U_2}\right)^{-1} \propto p^{-\alpha} \tag{4.105}$$

so that

$$p \propto t^{1/\alpha}. \tag{4.106}$$

On the other hand, the number of particles, N, decreases due to escape so that

$$N \propto p^{-3\chi_2} \propto t^{-3\chi_2/\alpha} \tag{4.107}$$

and these remaining particles occupy a region of width $L \propto \kappa/U \propto t$ around the shock. Thus the pressure:

$$P_C \propto \frac{N}{L} pv \propto t^{-3\chi_2/\alpha} t^{-1} t^{(3\gamma_C - 3)/\alpha}$$

$$= t^{3(\gamma_C - \chi_1)/\alpha - 1}. \tag{4.108}$$

The interesting aspect here is that the particle pressure can continue to increase after the injection has been switched off if $\alpha < 3(\gamma_C - \chi_1)$. This of course assumes the shock to be infinitesimally thin; if we broaden the transition the momentum gain decreases, the escape rate increases and at some finite thickness the pressure growth rate will be zero. This suggests that there may exist strong shock solutions where there is no injection or advection of fresh particles, but where an ever decreasing number of ever more energetic particles maintain a constant downstream pressure and a shock structure whose width increases linearly with time. If the particles are relativistic and the overall compression of the secularly broadening transition is close to the steady value of seven this requires $\alpha < 0.5$. Interestingly, the particles which escape form a power law spectrum $f \propto p^{\alpha - 4}$, so that this may be a way to get spectra flatter than 4. Any further discussion would involve us in too much speculation, but these simple arguments do suggest that both secular and periodic shock structures need to be investigated.

5. Concluding remarks

As stated in the introduction, and as I hope the presentation in § 2.3 has shown, the central idea of diffusive shock acceleration is simple and persuasive. Assuming that the accelerated particles could be treated as reactionless 'test particles' and their transport as diffusive we found that a steady parallel plane shock produced a distribution function which was a power law in momentum. In § 3 we then showed that, as long as the particles are fast and can return across the shock, the obliquity of the magnetic field is unimportant and that, at root because the acceleration is a consequence of Liouville's theorem, the sole determinant of the spectral slope is the compression of the plasma in the shock. However, the acceleration time scale and the spectral deviations induced by, for example, curvature of the shock, do depend on other factors, in particular the diffusion coefficient.

Unfortunately, the reaction of the accelerated particles cannot be neglected if they contribute significantly to the total pressure (as they must if the acceleration is efficient). One consequence of including recoil terms is that if the diffusion is supposed to occur through scattering off resonant Alfvén waves these rapidly become non-linear in all except weak shocks as shown in §§ 4.1 and 4.2.2. If we ignore this problem and assume, as in the rest of § 4, that diffusive transport remains a valid approximation the reaction effects would be reasonably well understood were the diffusion coefficient constant, or at least bounded. Unfortunately this is only a convenient fiction. In reality it must increase and, because of the high-energy divergence of the particle pressure that this would cause if the shock were steady, acceleration by strong shocks must be treated as a time-dependent problem. Nor, at the other end of the spectrum, does there exist a satisfactory theory for the injection of particles from the thermal background into the diffusive acceleration mechanism.

Until these gaps in our understanding are filled (and this should be possible with modern numerical and analytical methods) it is hard to pass any final judgment on diffusive shock acceleration. However, it is worth noting that the problems which afflict the theory arise mainly from the fact that as an accelerator the mechanism is too good; presumably Nature is better than we are at finding ways of keeping it in check, but it is hard to see how the basic mechanism could be completely suppressed. I remain optimistic that with the discovery of diffusive shock acceleration we have made a step towards understanding the seventy year old problem of the origin of cosmic radiation.

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