MHD instabilities of accretion disks and jets

– a new spectral theory of rotating plasmas –

Hans Goedbloed & Rony Kepppens

FOM-Institute for Plasma Physics 'Rijnhuizen'

Astronomical Institute, Utrecht University

Center for Plasma Astrophysics, K.U. Leuven

[*goedbloed@rijnh.nl*]

1. Introduction

- Fusion & astrophysical plasmas $\Rightarrow$ **Spectral theory of moving plasmas**
- Theme of new textbook on Advanced MHD $\Rightarrow$ **Stationary plasma flow**

2. New spectral theory

- Self-adjoint operators $\mathbf{G}$ and U $\Rightarrow$ **Real quadratic forms W and V**
- Energy flow in open system $\Rightarrow$ **Solution path in the complex ω plane**
- Oscillation theorem $\Rightarrow$ **Alternator monotonic on solution path**

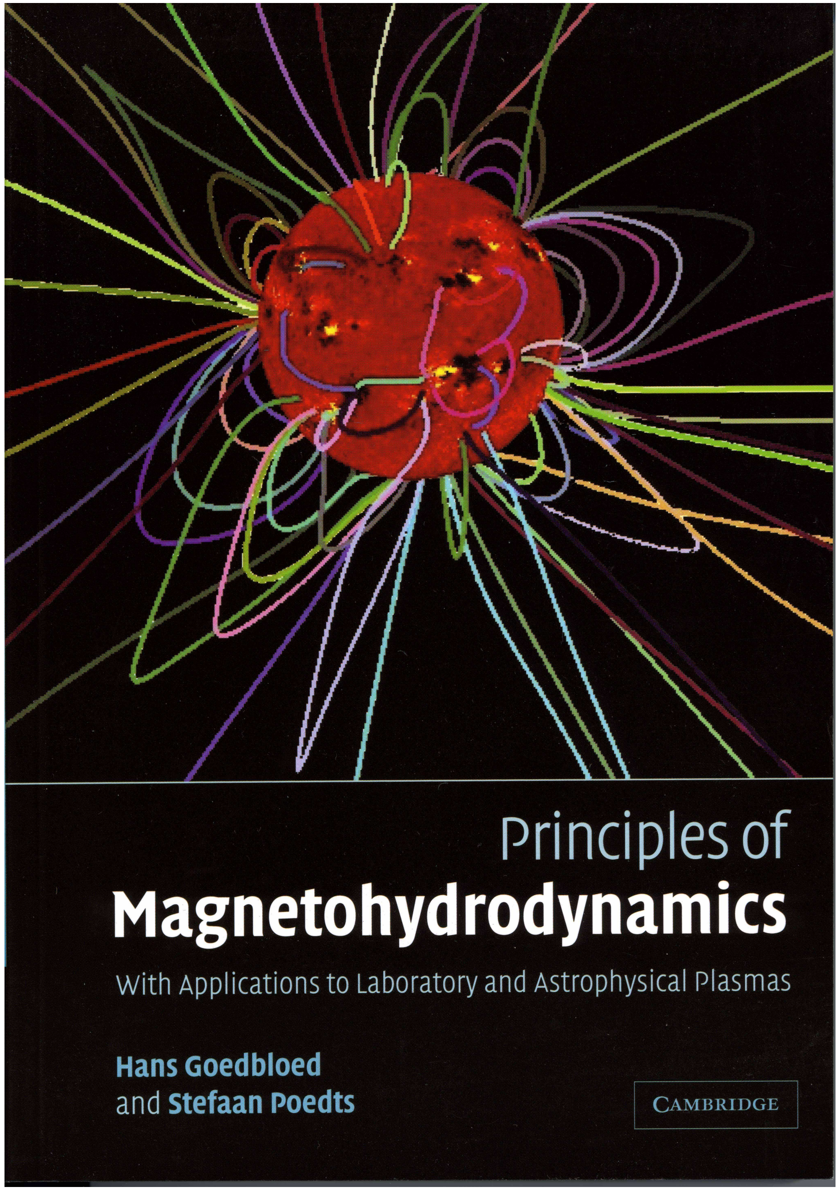
3. Applications

- **Spectral web for magneto-rotational and other instabilities**
- **Rotational stabilization of jets**

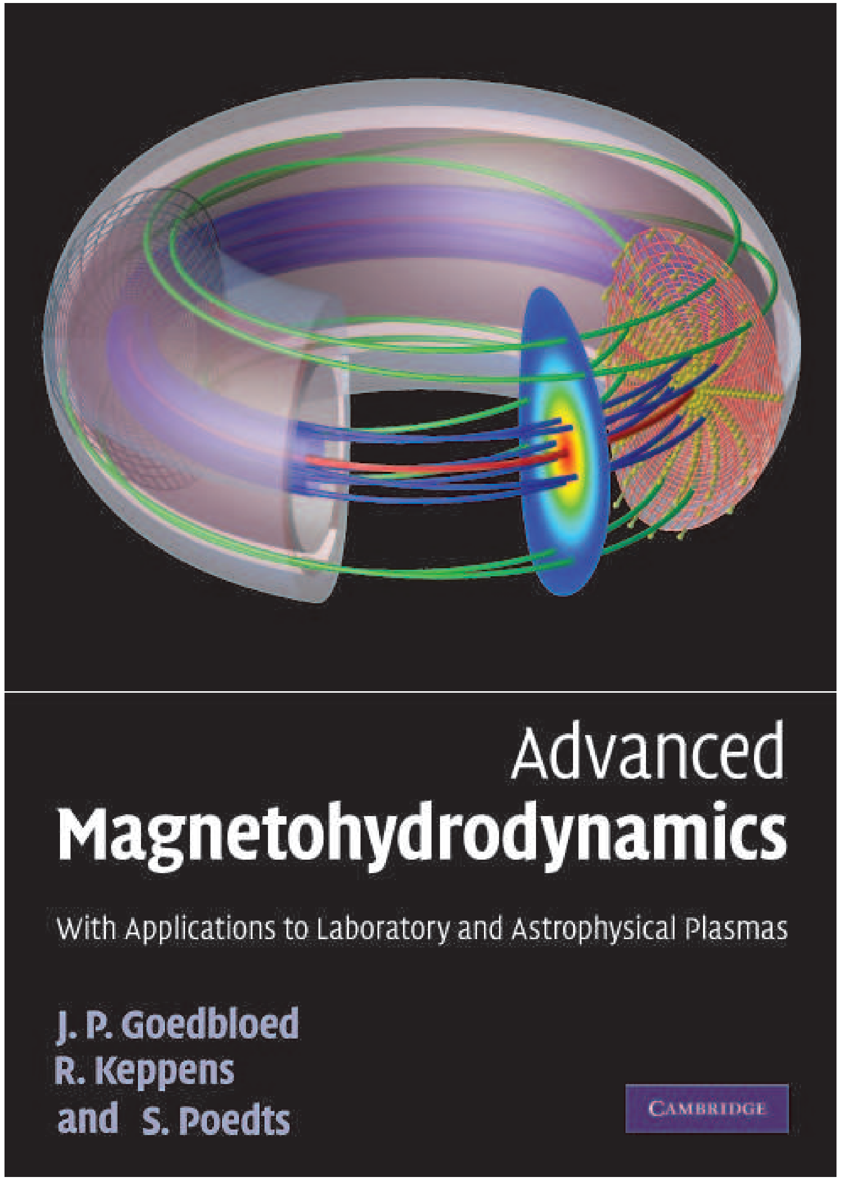
4. Summary

Two textbooks on Magnetohydrodynamics

Volume I (2004)



Volume II (2010)



Conservation laws & Scale independence

- **Ideal MHD equations** in terms of ρ, $\mathbf{v}$, p, $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,,$$

Conservation of mass

$$\rho \Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \Big) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,,$$

momentum

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,,$$

entropy

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,.$$

magnetic flux!

- **They are independent of length scale** (l_0), **density** (ρ_0) and **magnetic field** (B_0)

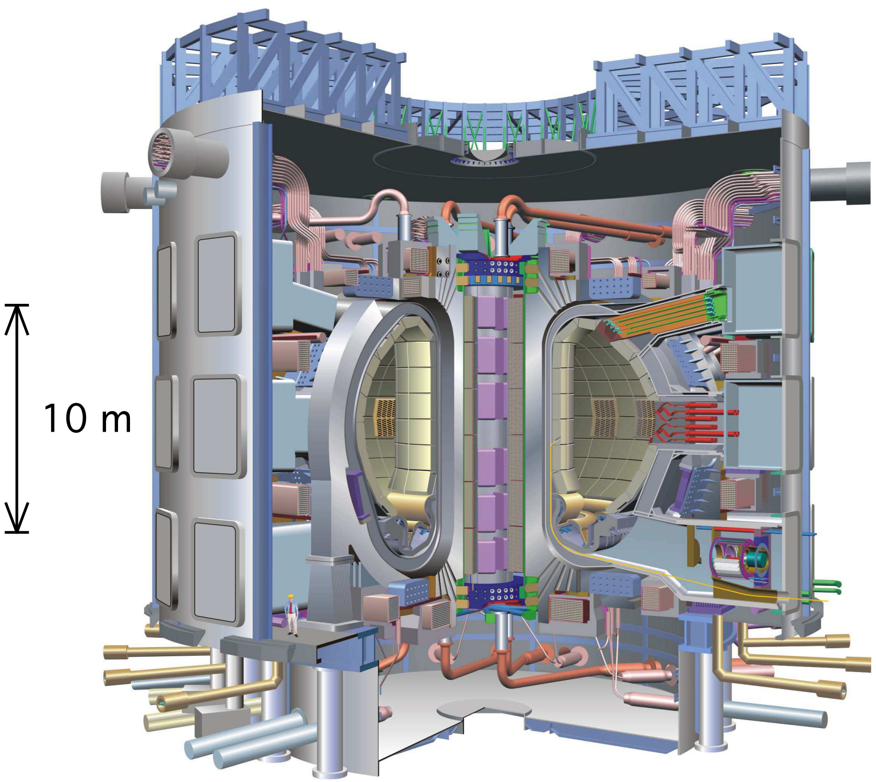
  $\Rightarrow$ **describe global dynamics of both laboratory and astrophysical plasmas!**

- **Of course, to be supplemented with appropriate boundary conditions.**

Magnetized plasma

is omni-present and described by magnetohydrodynamics

- Tokamak (Iter)



- Pinwheel Galaxy M101 (HST)

⇒ **Nuts and bolts** fix static plasma

⇒ **Gravity and rotation** fix moving plasma

Fusion plasmas

- **Energy principle for static plasmas (1957):** standard stability paradigm for more than 50 years ⇒ interchanges, kinks, peeling–ballooning, RWM **($\mathbf{k} \perp \mathbf{B}$).**
- **Modification for stationary plasmas (Frieman–Rotenberg, 1960):** known, but hardly investigated due to misnomer "non self-adjoint". Shear flow stabilizes some instabilities, but also drives new ones ⇒ **Kelvin–Helmholtz (KH)**.

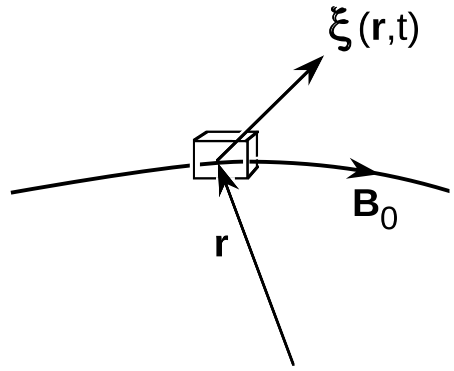
Astrophysical plasmas

- **Energy principle does not apply** since there are no static astrophysical plasmas.
- **Gravity and differential rotation establish equilibrium,** but also drive instabilities (violating tokamak "intuition": $\mathbf{k} \parallel \mathbf{B}$) ⇒ **Rayleigh–Taylor (RT), Parker, MRI, . . .**

MHD spectroscopy of stationary plasma flow

- **Urgent common theme** for laboratory and astrophysical plasma research.
- **Demands fundamentally different approach** from static flow, that can be based on two foundations laid 50 (Frieman–Rotenberg) and 100 (Hilbert) years ago.

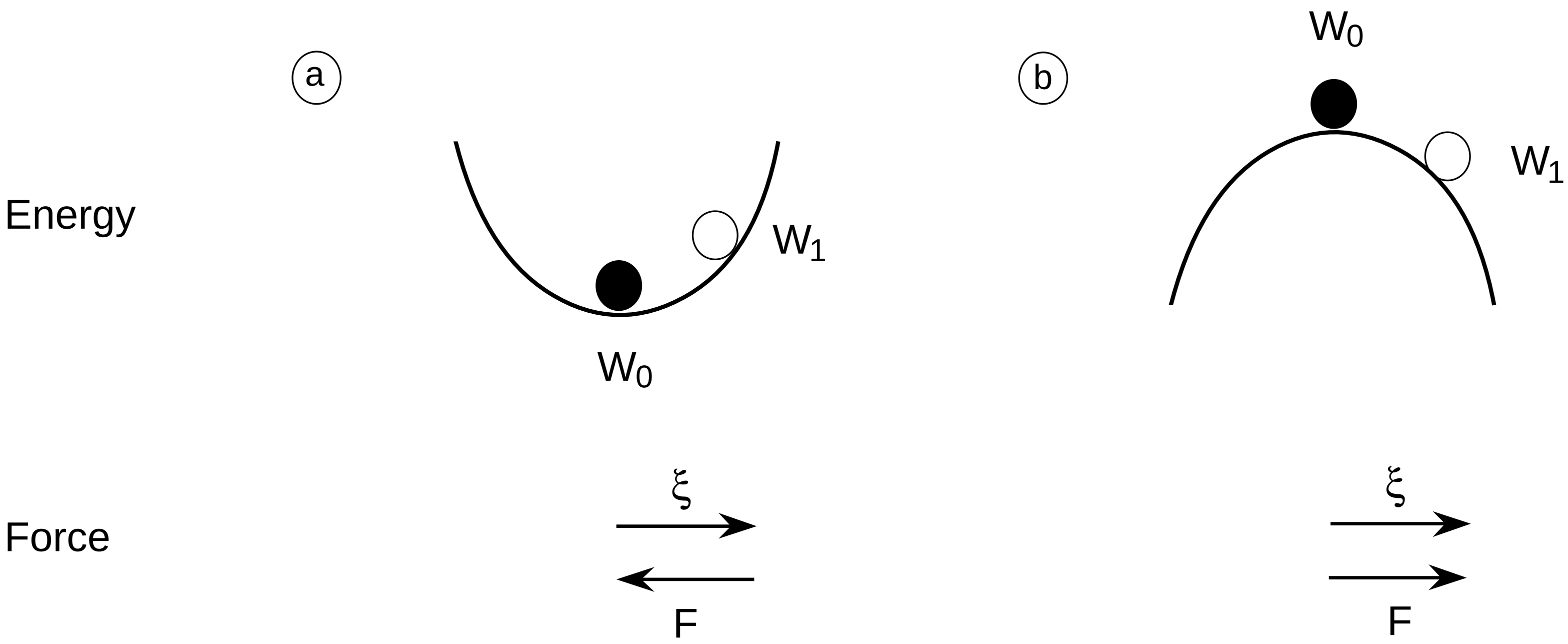
Displacement vector



⇒ Solves 3 of the 4 PDEs, so that only 'Newtons law' remains:

$$\mathbf{F}(\boldsymbol{\xi}) = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2 \boldsymbol{\xi}$$ (for normal modes $e^{i\omega t}$)

⇒ Energy: $W \equiv -\frac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV$ Hain *et al.* (1957), Bernstein *et al.* (1958)

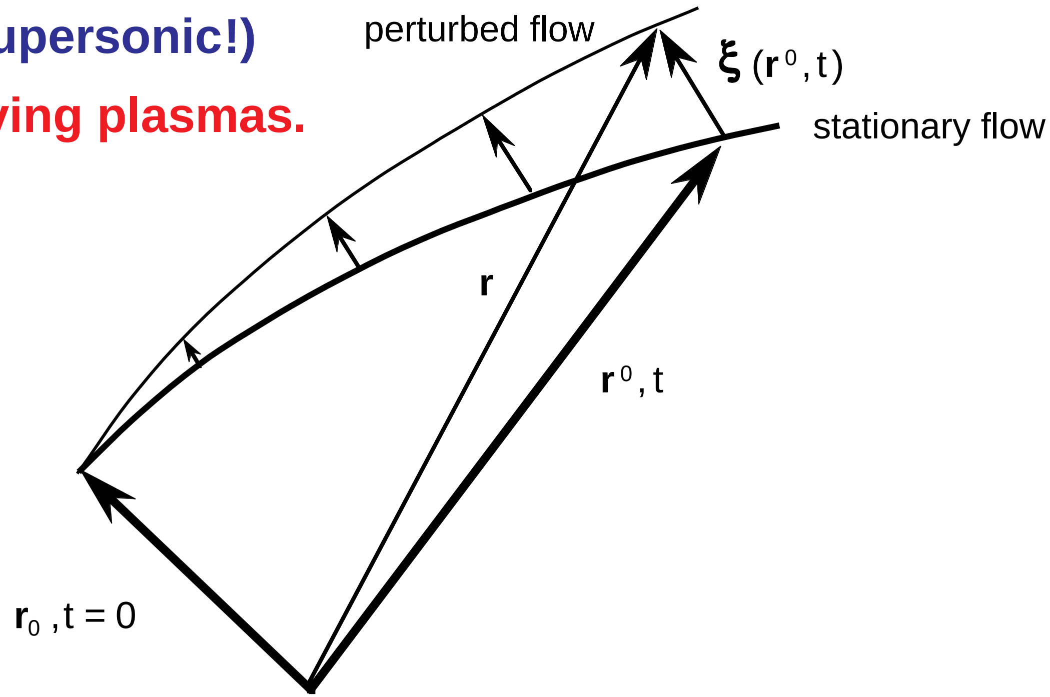


⇒ More than 50 years applied to tokamaks, and even to astrophysical plasmas!

Modified displacement

Since astrophysical (and also present tokamak) plasmas are not static at all (even supersonic!)

$\Rightarrow$ **Need MHD spectroscopy for moving plasmas.**



$\Rightarrow$ Again solves 3 of the 4 PDEs so that 'Newtons law' remains:

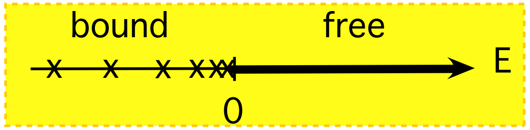
$$\boxed{\mathbf{G}(\boldsymbol{\xi}) - 2\rho \mathbf{v}\cdot\nabla \frac{\partial \boldsymbol{\xi}}{\partial t} - \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{G}(\boldsymbol{\xi}) - 2\omega U \boldsymbol{\xi} + \rho\omega^2 \boldsymbol{\xi} = 0}$$

Frieman–Rotenberg (1960)

$\mathbf{G}$: **generalized force operator,** $U \equiv -\mathrm{i}\rho\mathbf{v}\cdot\nabla$: **Doppler–Coriolis shift operator**

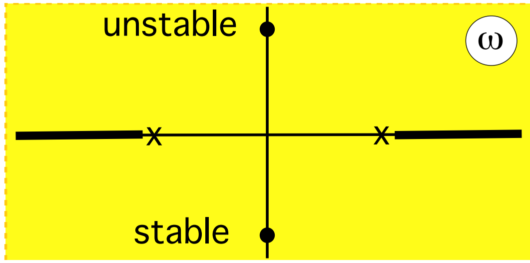
$\big($for plain waves $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$: $\; = \mathbf{k}\cdot\mathbf{v}_0\big)$

$$(1)\quad \mathbf{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t} = \mathbf{E}\Psi$$ (1926)



Quantum mechanics (atoms, molecules, condensed/living matter . . . everything?):

Hamiltonian H $\Rightarrow$ real EVs E $\longrightarrow$ stable solutions!

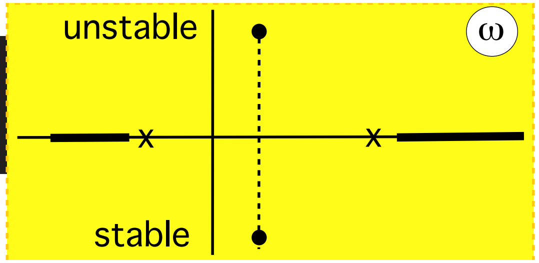


$$(2)\quad \mathbf{F}(\boldsymbol{\xi}) = \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2\boldsymbol{\xi}$$ (1957)

MHD of static plasmas (fusion only):

Force operator $\mathbf{F}$ $\Rightarrow$ real EVs ω^2 $\longrightarrow$ $\begin{cases} W > 0 \ (\omega \text{ real}) & \longrightarrow \text{stable waves} \\ W < 0 \ (\omega \text{ imag.}) & \longrightarrow \text{instabilities} \end{cases}$

$$(3)\quad \mathbf{G}(\boldsymbol{\xi}) - 2\rho\mathbf{v}\cdot\nabla\frac{\partial\boldsymbol{\xi}}{\partial t} - \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = \mathbf{G}(\boldsymbol{\xi}) - 2\omega U\boldsymbol{\xi} + \rho\omega^2\boldsymbol{\xi} = 0$$



MHD of moving plasmas (fusion/astrophysical . . . cosmic):

Generalized force $\mathbf{G}$ and Doppler–Coriolis $U \equiv -\mathrm{i}\rho\mathbf{v}\cdot\nabla$

$\Rightarrow$ EVs $\begin{cases} \omega \text{ real} & \longrightarrow \text{stable (undamped) waves} \\ \omega \text{ complex} & \longrightarrow \text{instabilities/damped waves} \end{cases}$

How to compute them?

Obstacle

- Problems (1) & (2): extensively studied ($\sim$ ten thousands of papers).
- Problem (3): hardly studied ($\sim$ hundreds of papers), due to widely held belief that **"the problem is non-self-adjoint".**
- How come? **Energy is conserved, and both $\mathbf{G}$ and U are self-adjoint!**

Quadratic forms

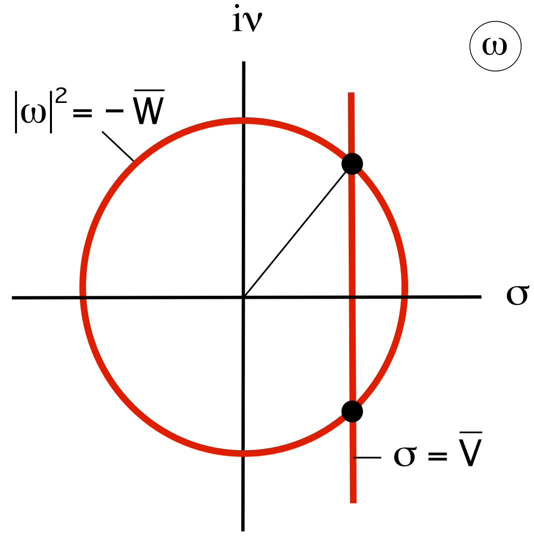
- Inner product and norm: $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \frac{1}{2} \int \rho \, \boldsymbol{\xi}^* \cdot \boldsymbol{\eta} \, dV \,, \quad \boxed{I[\boldsymbol{\xi}] \equiv \|\boldsymbol{\xi}\|^2 \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle < \infty}.$
- **Operators are self-adjoint:**

$$\langle \boldsymbol{\eta}, \rho^{-1} U \boldsymbol{\xi} \rangle = \langle \rho^{-1} U \boldsymbol{\eta}, \boldsymbol{\xi} \rangle \quad \Rightarrow \quad \textit{real} \quad \boxed{V \equiv \tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot U \boldsymbol{\xi} \, dV} \quad \text{(Doppler shift)} \,,$$

$$\langle \boldsymbol{\eta}, \rho^{-1} \mathbf{G}(\boldsymbol{\xi}) \rangle = \langle \rho^{-1} \mathbf{G}(\boldsymbol{\eta}), \boldsymbol{\xi} \rangle \quad \Rightarrow \quad \textit{real} \quad \boxed{W \equiv -\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{G}(\boldsymbol{\xi}) \, dV} \quad \text{(energy)} \,.$$

- **But eigenvalue problem (3) is nonlinear:**

$$\boxed{\omega^2 - 2\overline{V}\omega - \overline{W} = 0}\,, \qquad \overline{V} \equiv V/I \equiv \langle \rho^{-1} U \rangle \,, \quad \overline{W} \equiv W/I \equiv \langle -\rho^{-1} \mathbf{G} \rangle \,.$$



- 'Solutions' of the quadratic, with $\omega \equiv \sigma + \mathrm{i}\nu$:

$$\begin{cases} \sigma = \overline{V} \pm \sqrt{\overline{W} + \overline{V}^2}\,, \quad \nu = 0 & \text{(stable waves)} \\ \boxed{\sigma = \overline{V}\,, \quad \nu = \pm\sqrt{-\overline{W} - \overline{V}^2}} & \text{(instabilities)} \end{cases}$$

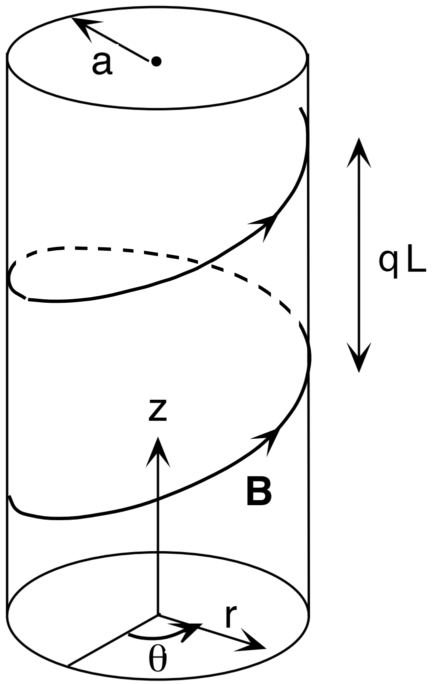
  This expression determines stability and yields picture of where actual eigenvalues are located $\Rightarrow$

- Would also yield a computational procedure if we knew:
  How to compute solution averages before eigenvalue (EV) is obtained?

- Recall static ('linear') eigenvalue problem:
  (1) $\mathbf{F}$ self-adjoint $\Rightarrow \omega^2$ real $\Rightarrow$ **EVs ω lie on the real and imaginary axes.**
  (2) EVs monotonic with number of zeros of ξ (Goedbloed–Sakanaka, 1974).

- In stationary problem, Doppler–Coriolis shift $\overline{V}$ moves EVs off the imaginary axis:
  $\Rightarrow$ **(1) Solution path $\equiv$ unknown curve on which the EVs are located?**
  $\Rightarrow$ **(2) Monotonicity property of EVs on the solution path?**

ROC



Arbitrary $\rho(r)$, $p(r)$, $v_\theta(r)$, $v_z(r)$, $B_\theta(r)$, $B_z(r)$,

but satisfying the equilibrium condition,

$$(p + \tfrac{1}{2}B^2)' = (\rho v_\theta^2 - B_\theta^2)/r - \rho\Phi'_{\rm gr}\,.$$

Apply to two generic astrohysical problems:

(1) Accretion disk model, thin slice Δz:

annulus Δr (M_* at $r = 0$), $k\Delta z \gg 1$;

(2) Rotating jet of finite length L:

plasma + 'vacuum', $\Phi_{\rm gr} = 0\,,\; k = n\pi/L$.

Reduction of Frieman–Rotenberg equation, with $\chi \equiv r\xi_r = \hat{\chi}(r)\mathrm{e}^{\mathrm{i}(m\theta + kz - \omega t)}$:

$$\frac{d}{dr}\left[\frac{N}{D}\frac{d\chi}{dr}\right] + \left[A + \frac{B}{D} + \left\{\frac{C}{D}\right\}'\right]\chi = 0\,,$$

or

$$N\frac{d}{dr}\begin{pmatrix}\chi\\ \Pi\end{pmatrix} + \begin{pmatrix}C & D\\ E & -C\end{pmatrix}\begin{pmatrix}\chi\\ \Pi\end{pmatrix} = 0\,,$$

where $N = N(r;\widetilde{\omega})$, with $\widetilde{\omega} \equiv \omega - \mathbf{k}_0\cdot\mathbf{v}$, and Π is the total pressure perturbation.

BCs:

$\chi(r_1) = 0$ (left)

$\chi(r_2) = 0$ (right)

$\Rightarrow$ **Eigenvalue problem.**

Spectral properties

For plane slab: ODE similar to static case (Goedbloed, 1971), but ω is replaced by the **Doppler-shifted frequency** in co-moving layers:

$$\omega \quad \rightarrow \quad \widetilde{\omega}(x) \equiv \omega - \Omega_0(x)\,, \qquad \Omega_0 \equiv \mathbf{k}_0 \cdot \mathbf{v}(x)\,.$$

For cylinder: Hain–Lüst eq. (1958), generalized for rotation by Bondeson *et al.* (1987), and for gravitating thin disk with MRI by Keppens *et al.* (2002).

$\Omega_0 = m v_\theta / r + k v_z\,,$ **and Coriolis terms** $\sim v_\theta / r$!!

Previous results:

- HD $\Rightarrow$ **flow continuum** $\{\Omega_0(x)\}$, discovered by Case (1960).
- MHD $\Rightarrow$ contrary to prolonged belief, **no flow continuum!** (Goedbloed *et al.*, 2004).
- Instead, three static MHD continua split into **six Doppler-shifted continua:**

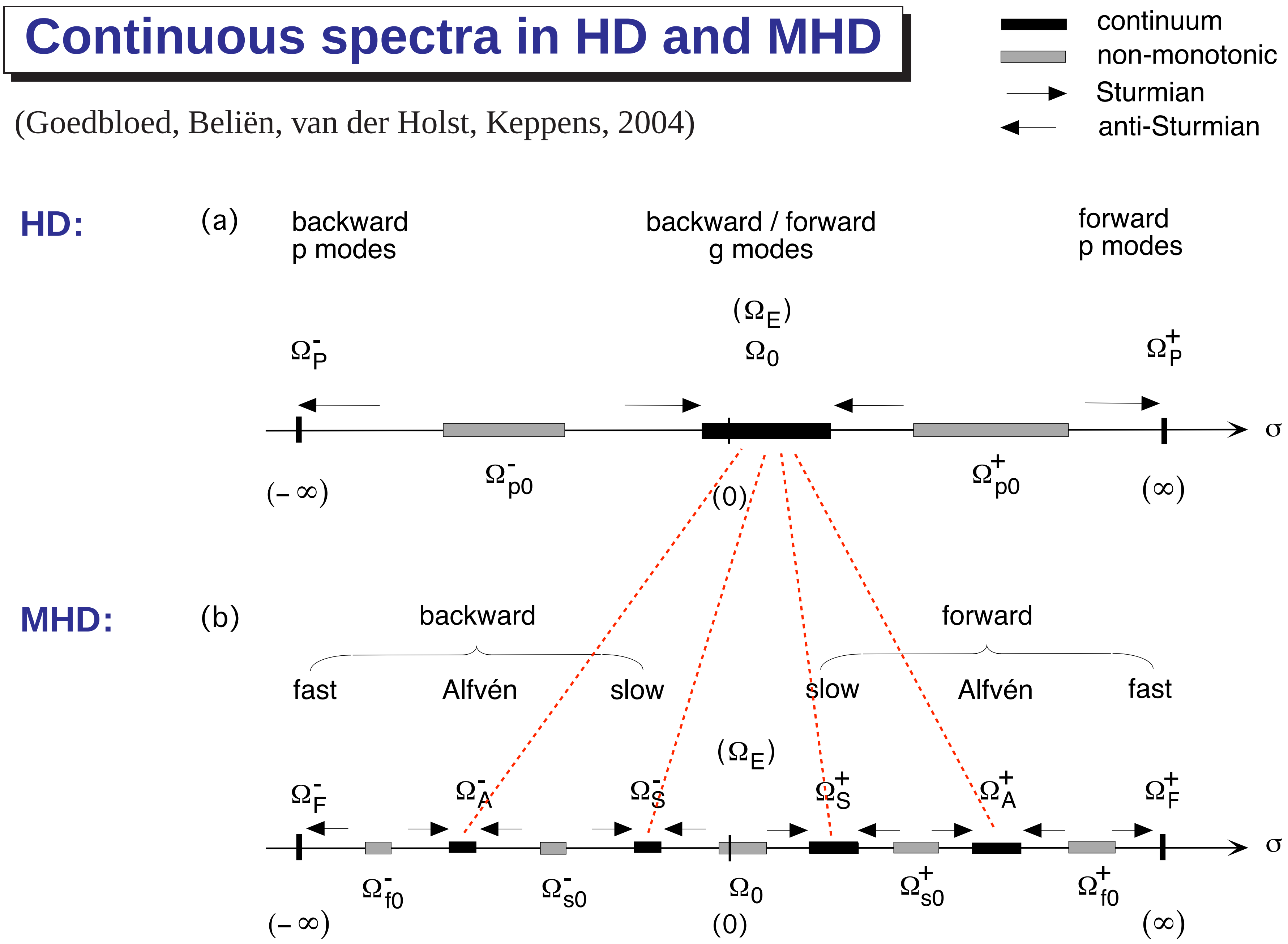
  $\Omega_A^\pm \equiv \Omega_0(x) \pm \omega_A(x)$ (Alfvén), $\Omega_S^\pm \equiv \Omega_0(x) \pm \omega_S(x)$ (slow), $\Omega_F^\pm \equiv \pm\infty$ (fast).

  Flow continuum is obtained in the limit $\mathbf{B} \rightarrow 0$.

$\Rightarrow$ **How is the full (complex) spectrum connected to this (real) structure?**

Continuous spectra in HD and MHD

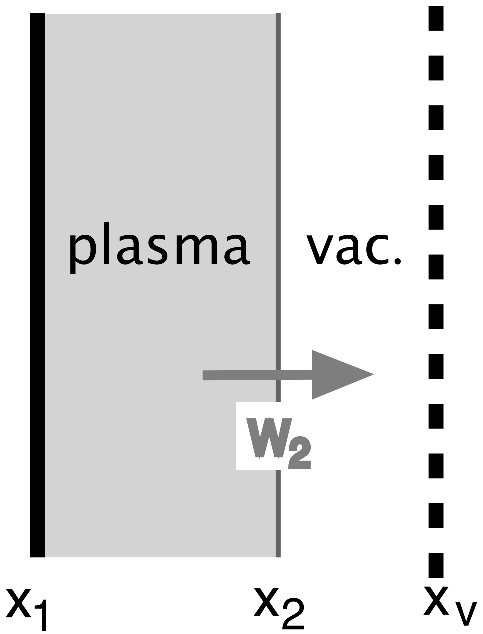
(Goedbloed, Beliën, van der Holst, Keppens, 2004)



Consider open system

(a) **Spectral differential equation** can be solved accurately for arbitrary complex ω.
$\Rightarrow$ **No problem!**

(b) Actual problem is searching in the complex ω-plane for the eigenvalues.
$\Rightarrow$ **Temporarily, drop that part by removing one of the boundaries!**



Keep: $\xi(x_1) = 0$ (left BC)

Drop: $\xi(x_2) = 0$ (right BC).

- To get harmonic time dependence $\exp(-\mathrm{i}\omega t)$, **energy has to be injected or extracted at x_2.** This is represented by imaginary part of energy, which we demand to vanish:

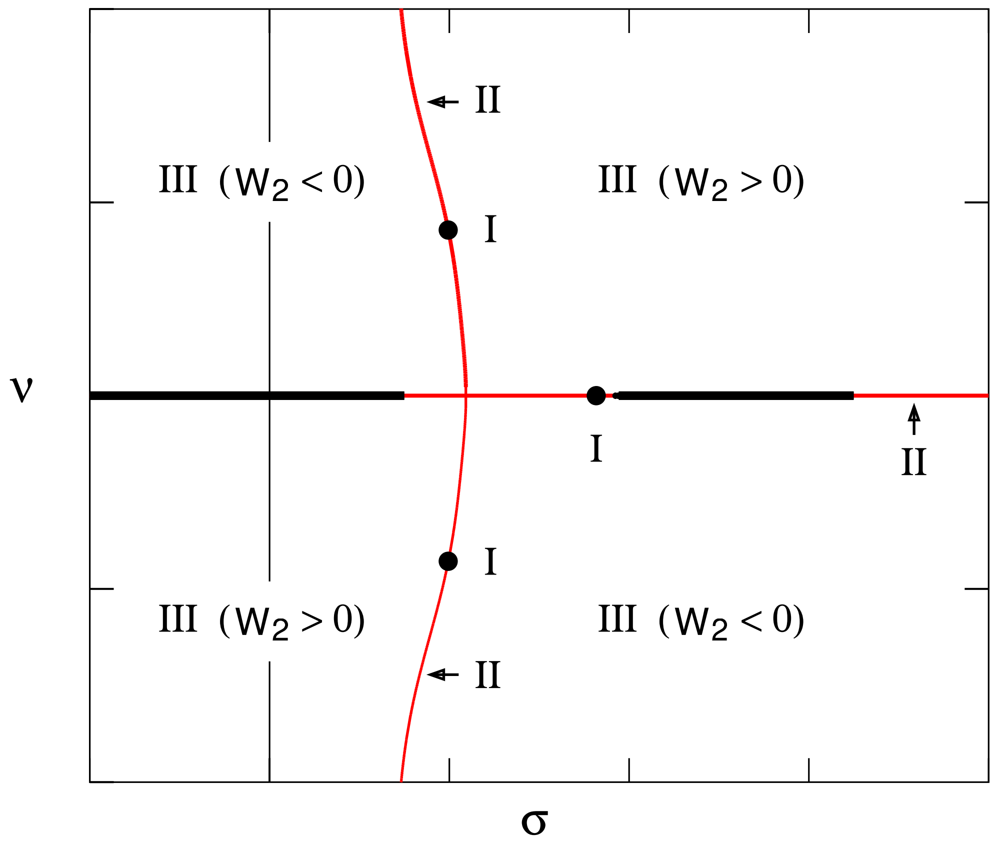
$$W_2 \equiv \mathrm{Im}(W) = 0 \quad \Rightarrow \quad \textbf{solution path!}$$

- Required expression follows directly from proof of **self-adjointness of the force operator $\mathbf{G}$.**

- **Eigenvalues have to lie on this path.**

Complex omega-plane

Problem solvable for arbitrary complex ω, but energy is complex: $W = W_1 + \mathrm{i}W_2$.



$\Rightarrow$ **Three BVPs:**

I – Eigenvalues (closed),
wall on the plasma:
$W_2 = 0\,, \quad \xi(x_2) = 0\,;$

II – Solution path (closed), $\Leftarrow$
e.g. variable vacuum layer:
$W_2 = 0\,, \quad \xi(x_\mathrm{v}) = 0\,;$

III – Arbitrary complex ω (open),
external excitation:
$W_2 \neq 0\,.$

Solution of the quadratic $\omega^2 - 2\overline{V}\omega - \overline{W} = 0 \quad \Rightarrow \quad \boxed{\sigma = \overline{V}\,, \quad \text{iff} \quad W_2 = 0}$

$\Rightarrow$ **average Doppler shifted real part of frequency vanishes iff system is closed.**

Solution path

- **Pre-self-adjointness relation for $\mathbf{G}$**, with $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ not yet satisfying right BC:

$$\int [\boldsymbol{\eta}^* \cdot \mathbf{G}(\boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \mathbf{G}(\boldsymbol{\eta}^*)]\, dV \overset{\text{(Gauss)}}{=} - \int [\eta^* \Pi(\boldsymbol{\xi}) - \xi\, \Pi(\boldsymbol{\eta}^*)]\, dS \quad \big(= 0 \text{ if BC}\big)\,.$$

Skip last step! Choosing $\boldsymbol{\eta} \equiv \boldsymbol{\xi}^*$ yields easily computable expression for W_2:

$$W_2 = \tfrac{1}{4}\mathrm{i} \int [\boldsymbol{\xi}^* \cdot \mathbf{G}(\boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \mathbf{G}(\boldsymbol{\xi}^*)]\, dV = \tfrac{1}{2} \int (\xi_1 \Pi_2 - \xi_2 \Pi_1)\, dS$$

$$\Rightarrow \boxed{W_2[\boldsymbol{\xi}(\mathbf{r};\omega)] = 0} \quad \Rightarrow$$ **path $\mathcal{P}_\mathrm{u}$ of unstable solutions.**

- Equivalently, **self-adjointness of U yields mapping of ω-plane onto itself,**

$$Q(\omega) \equiv \omega - \overline{V}[\boldsymbol{\xi}(\mathbf{r};\omega)] \equiv \omega - \frac{\int \boldsymbol{\xi}^* \cdot U \boldsymbol{\xi}\, dV}{\int \rho |\boldsymbol{\xi}|^2\, dV}\,,$$

which provides both solution paths:

$$\begin{cases} \mathrm{Im}\, Q \equiv \nu = 0 & \Rightarrow \\ \mathrm{Re}\, Q \equiv \boxed{\widetilde{\sigma} \equiv \sigma - \overline{V}\,[\boldsymbol{\xi}(\mathbf{r};\omega)] = 0} & \Rightarrow \end{cases}$$

path $\mathcal{P}_\mathrm{s}$ of stable solutions,

path $\mathcal{P}_\mathrm{u}$ of unstable solutions.

Oscillation theorems and alternator

- Once solution path is determined, EVs on it are found by imposing the missing BC. But how does one move from one EV to the next?

- **Oscillation theorem $\mathcal{R}$ for stable waves:** **Counting nodes of the real function ξ** yields Sturm–Liouville monotonicity (as static case: Goedbloed–Sakanaka 1974).

- **Instabilities:** **On the solution path, the alternating ratio $\mathrm{R} \equiv \xi / \Pi$ is real:**

$$W_2 = \tfrac{1}{2}\left[\xi_1\Pi_2 - \xi_2\Pi_1\right]_{x_2} = 0 \quad \Rightarrow \quad \boxed{R \equiv \frac{\xi(x_2)}{\Pi(x_2)} = \frac{\xi_1(x_2)}{\Pi_1(x_2)} = \frac{\xi_2(x_2)}{\Pi_2(x_2)}},$$

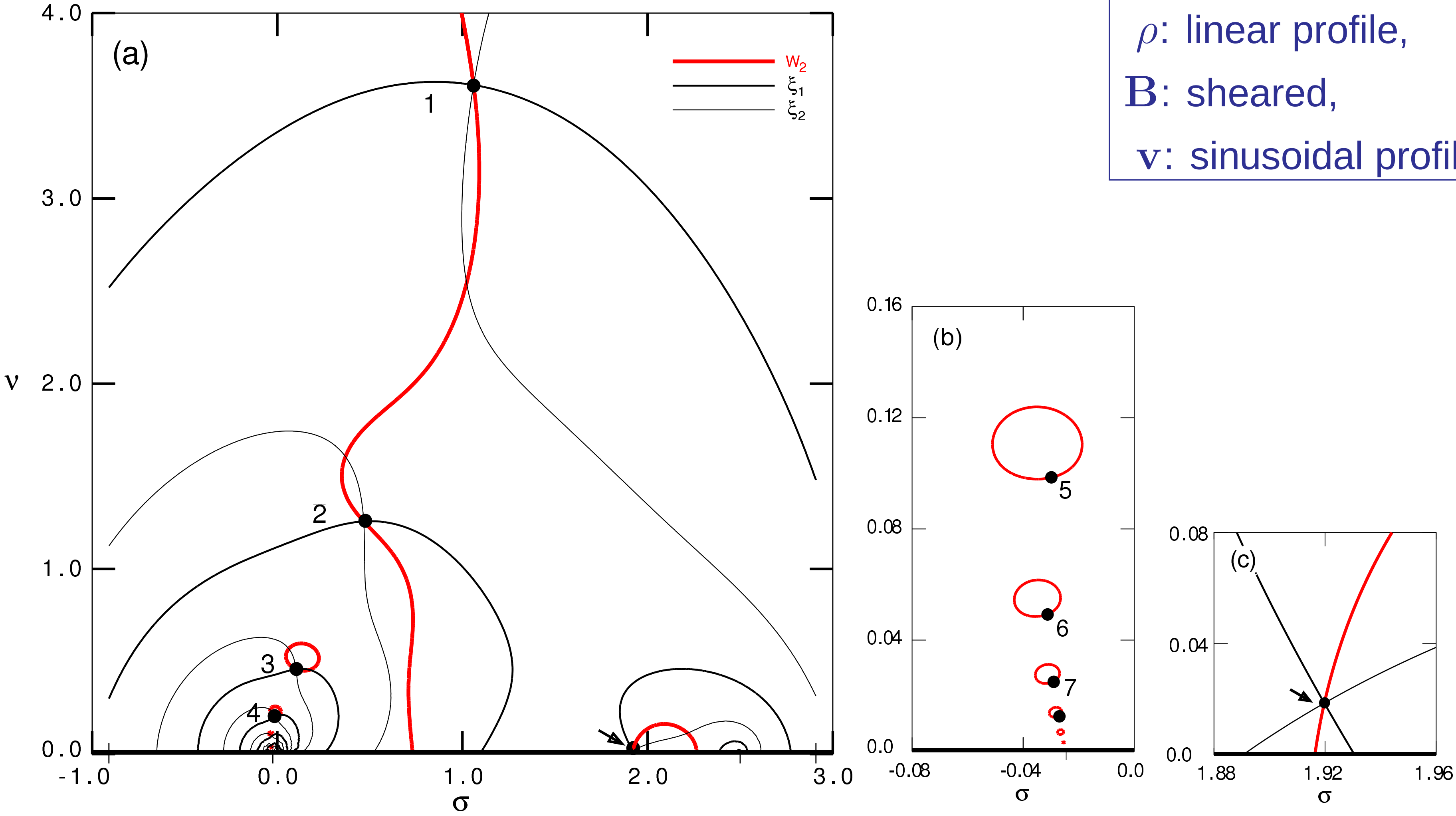
$$\boxed{R_1 = 0} \quad \Rightarrow \quad \textbf{Eigenvalues.}$$

- $\Rightarrow$ **Oscillation theorem $\mathcal{C}$ for instabilities** [proof exploits quadratic forms]**:**
  The alternator $\mathrm{R} \equiv \xi_e / \Pi_e$ is real and monotonic along the solution path in between the zeros of Π_e separating the eigenvalues.

- **Now, we are in business!**

Solution path

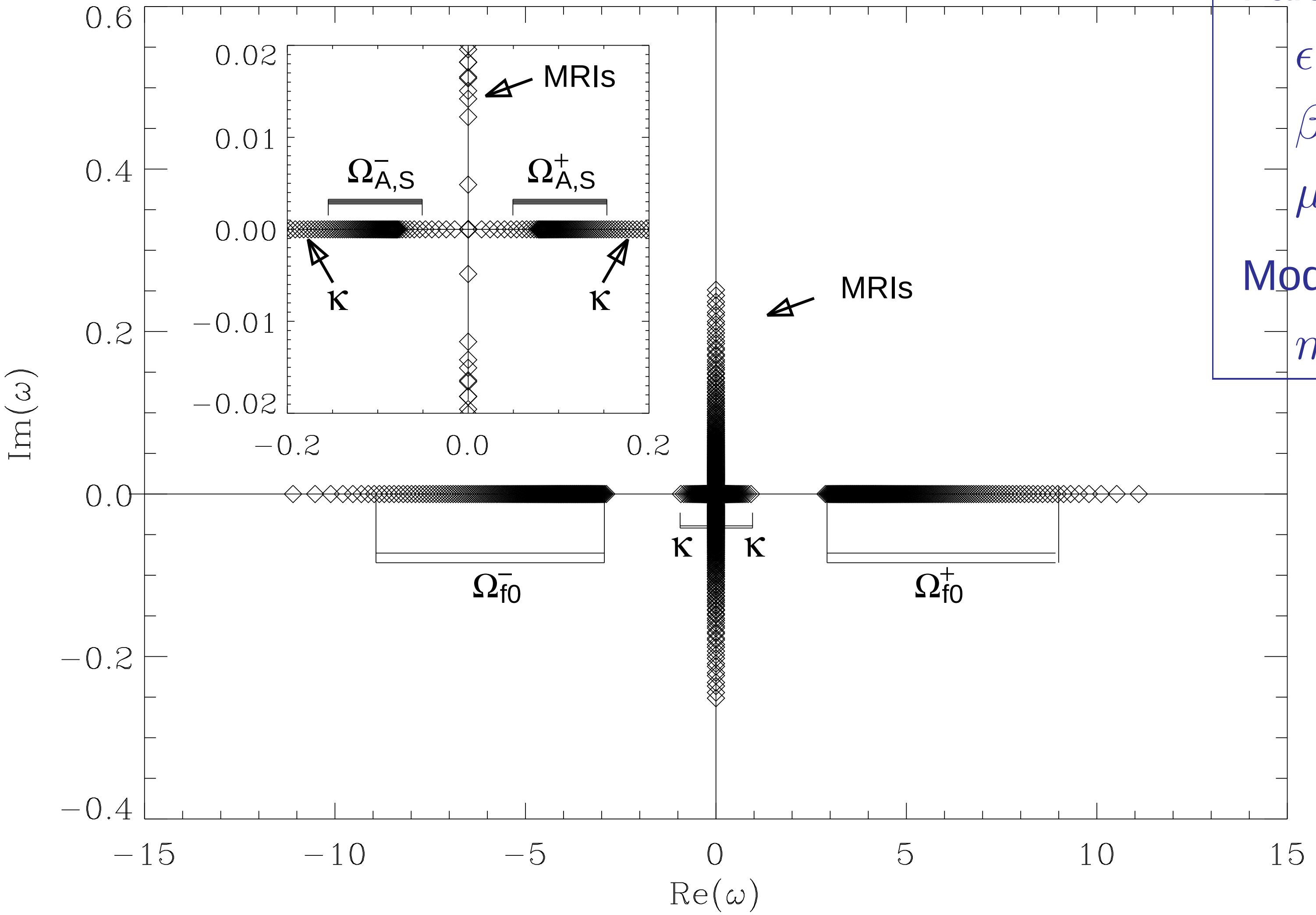
Plane gravitating slab
ρ: linear profile,
$\mathbf{B}$: sheared,
$\mathbf{v}$: sinusoidal profile.



⇒ Infinite sequence RT modes on ever smaller closed loops, one isolated KH mode.

Full spectrum (LEDA–FLOW)

[Keppens, Casse, Goedbloed, ApJL (2002)]



Standard equilibrium:

$\rho = r^{-3/2}, \; v_\theta \sim r^{-1/2}\,;$

Parameters:

$\epsilon \equiv \sqrt{p} \quad = 0.1\,,$

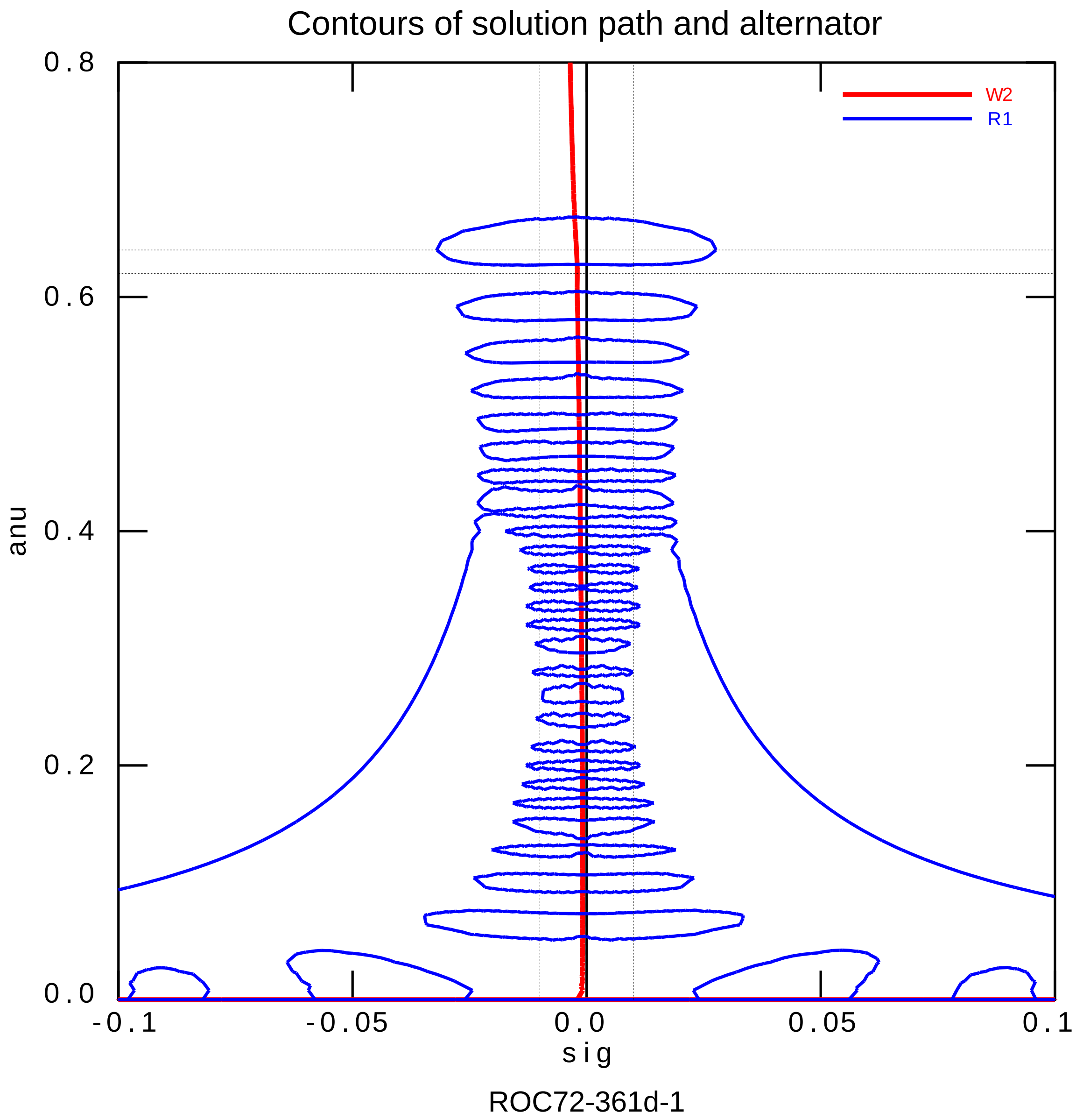
$\beta \equiv 2p/B^2 = 2000\,,$

$\mu \equiv B_\theta/B_z = 1\,;$

Mode numbers:

$m = 0\,, \quad k = 50$

Spectral web (ROC)



Equilibrium:

$$\epsilon = 0.1\,, \quad \beta = 100\,, \quad \mu = 1\,;$$

Mode numbers:

$$m = 0\,, \quad k = 50\,.$$

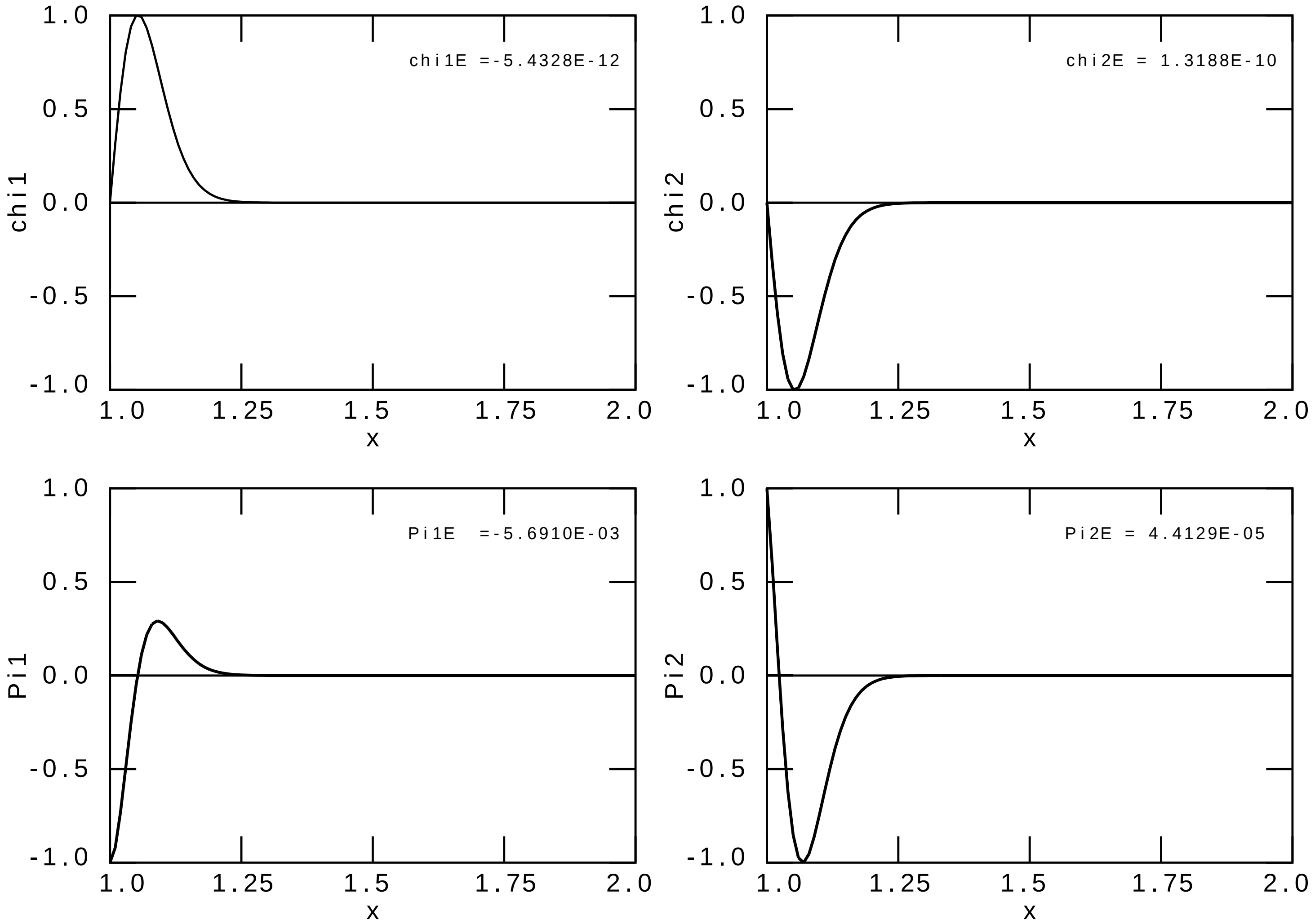
Solution path is not along imaginary axis:

$$\sigma = \overline{V} \neq 0$$

Alternator loops give genuine $(\xi_1 = \xi_2 = 0)$ **& false** $(\Pi_1 = \Pi_2 = 0)$ **eigenvalues.**

Fastest growing mode

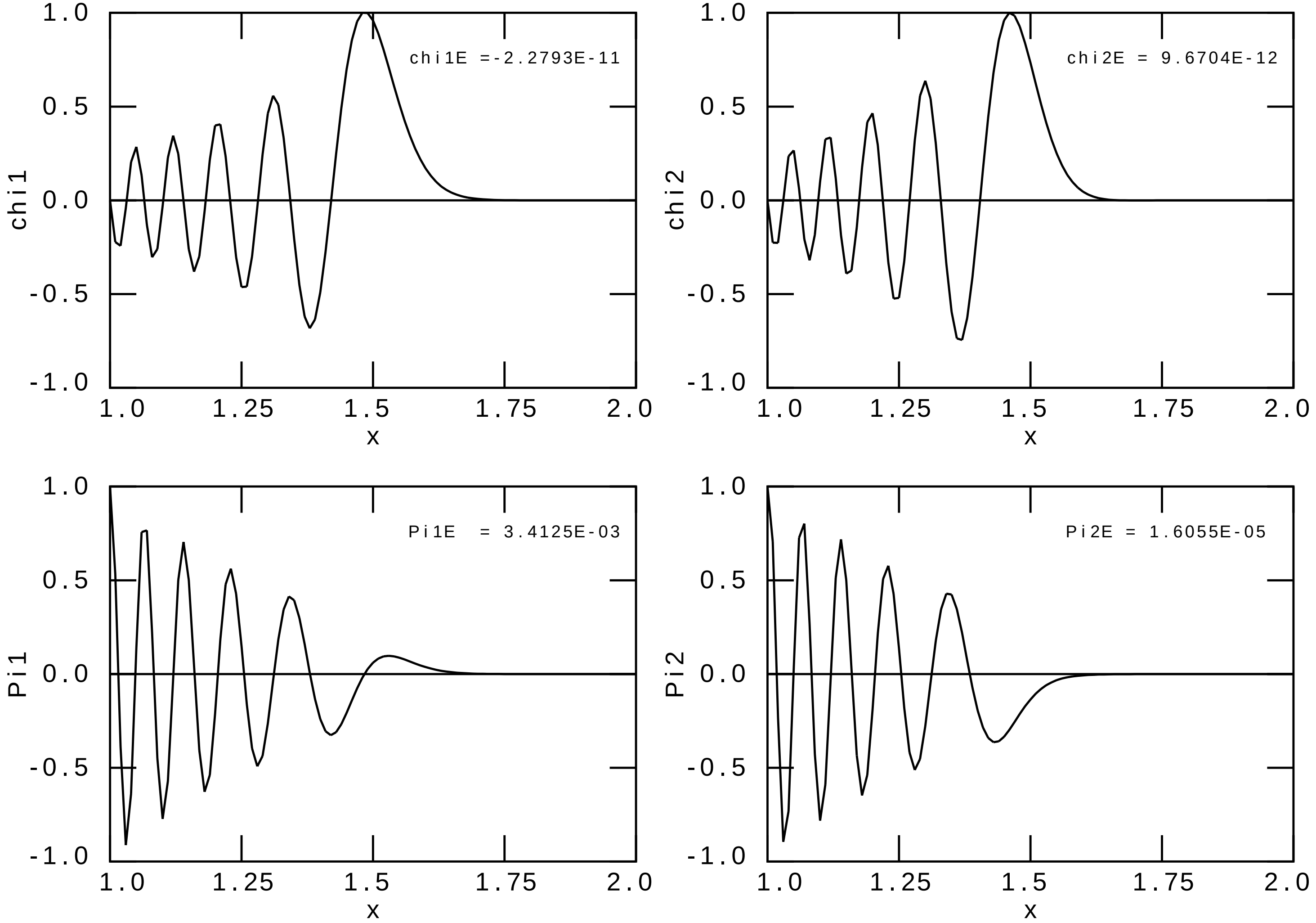
$$\sigma = -2.031 \times 10^{-3}\,,\ \nu = 0.6277$$



ROC72-361d-2

One of the cluster modes

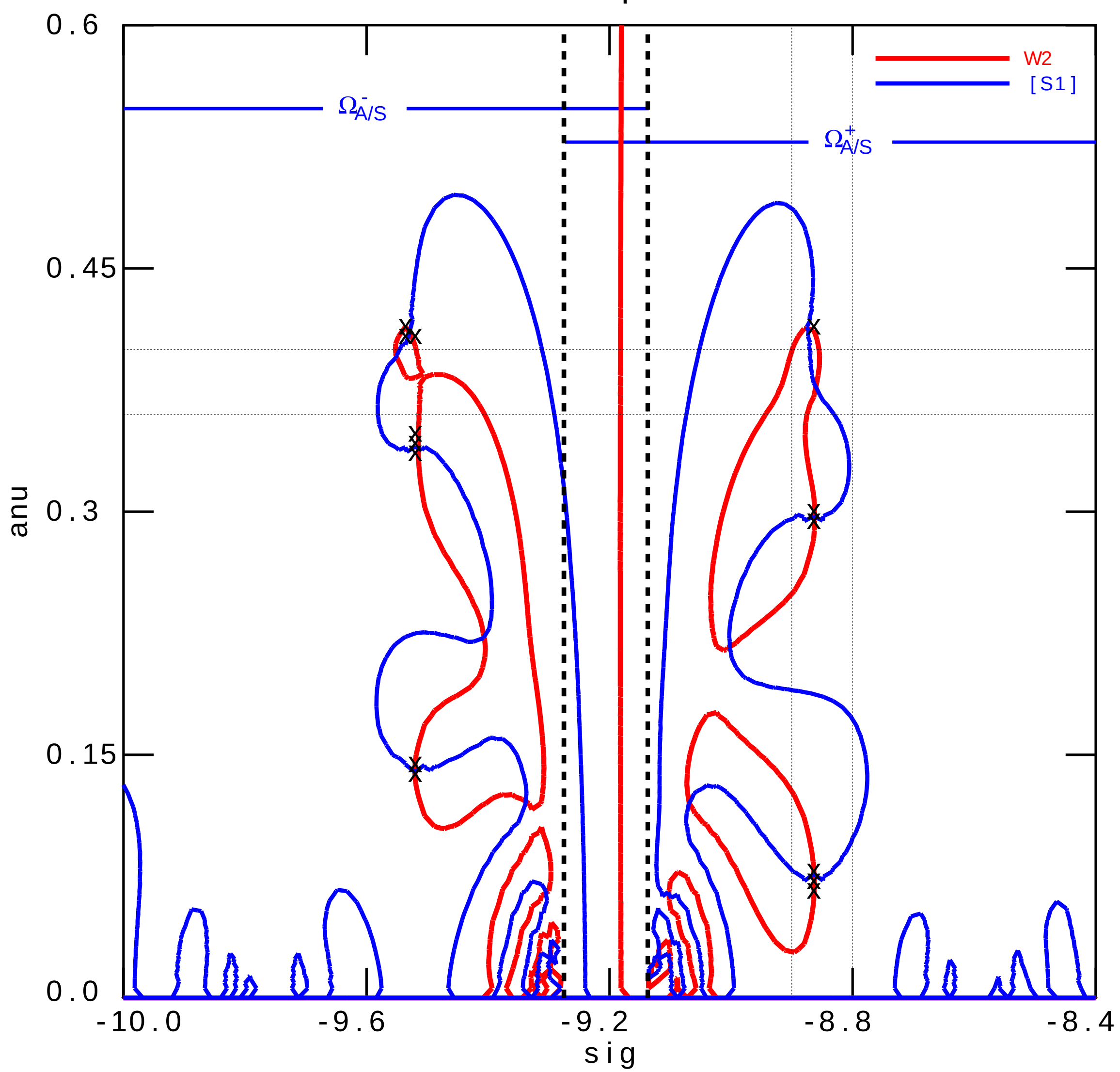
$$\sigma = -1.287 \times 10^{-3}\,,\ \nu = 0.3861$$



ROC73-361g

Spectral web (ROC)

Contours of solution path and alternator



ROC72-350-1

Equilibrium:

$$\epsilon = 0.1\,, \quad \beta = 100\,,$$
$$\mu = 1\,;$$

Mode numbers:

$$m = 10\,, \quad k = 50$$
$$\Downarrow$$

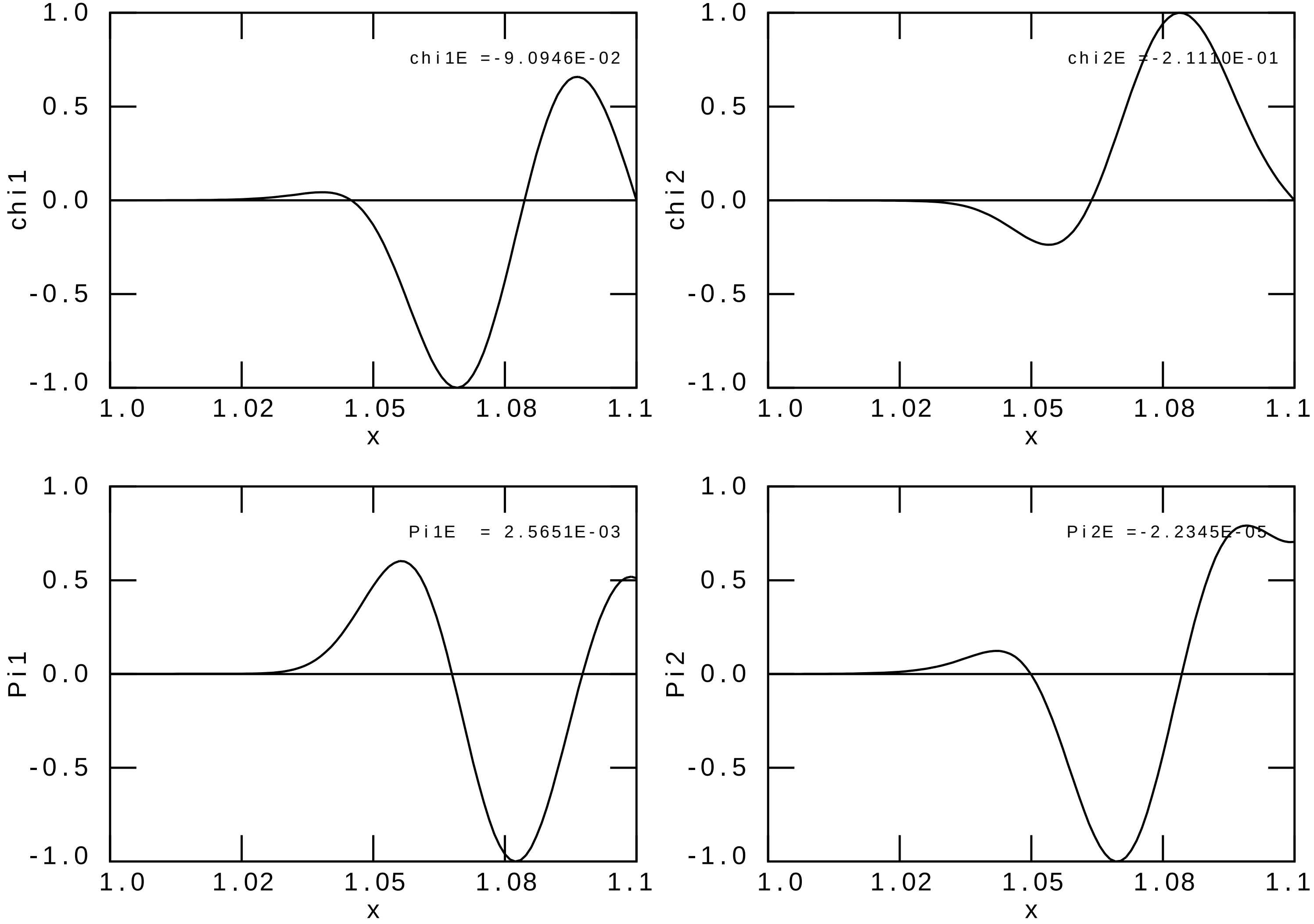
Overlapping continua:

$\Omega^+_{\mathrm{A,S}}$ and $\Omega^-_{\mathrm{A,S}}$.

Both, solution path and alternator form loops!

Fastest growing mode

$$\sigma = -8.860 \times 10^{-3}\,,\ \ \nu = 0.3753$$



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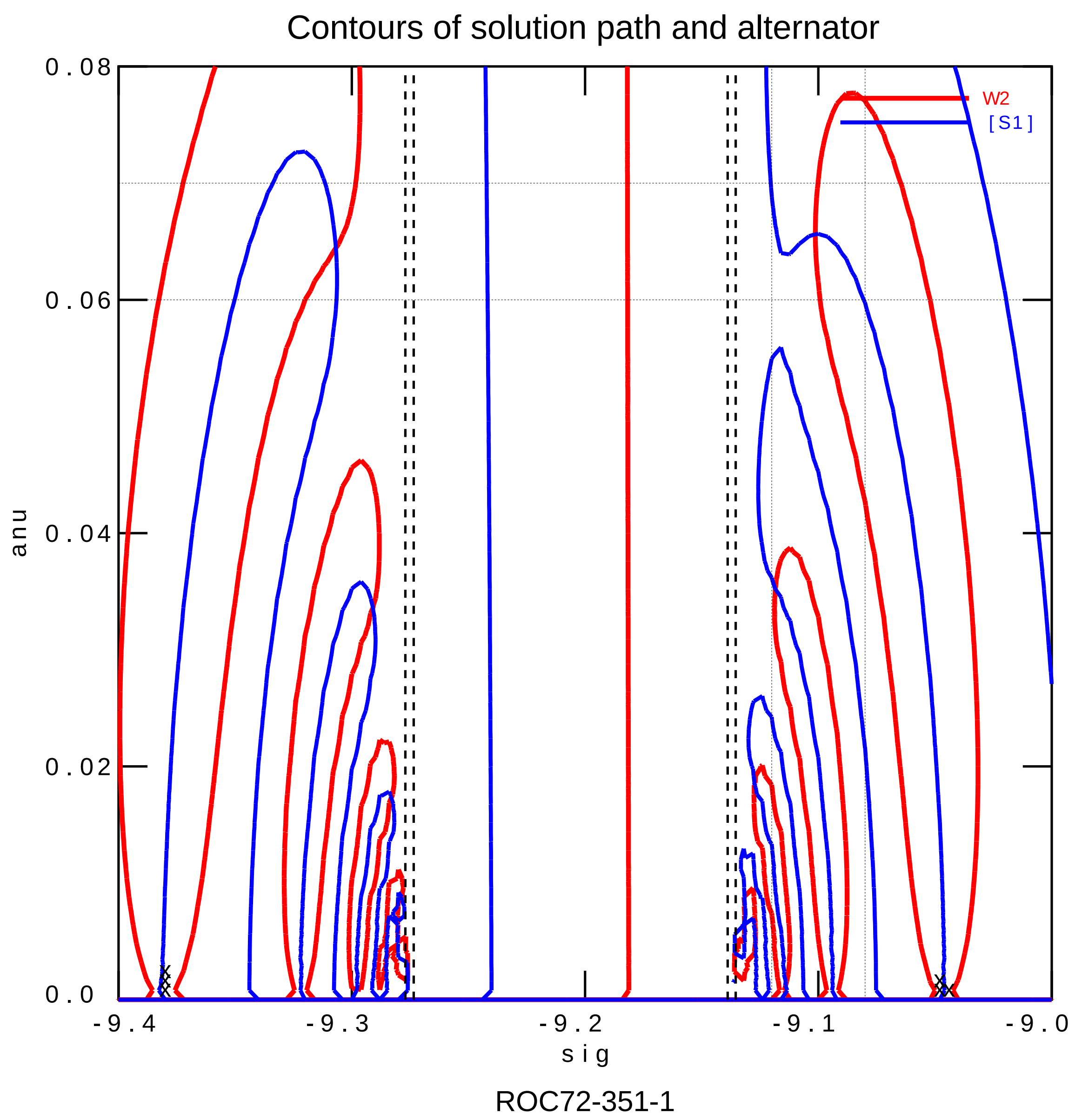
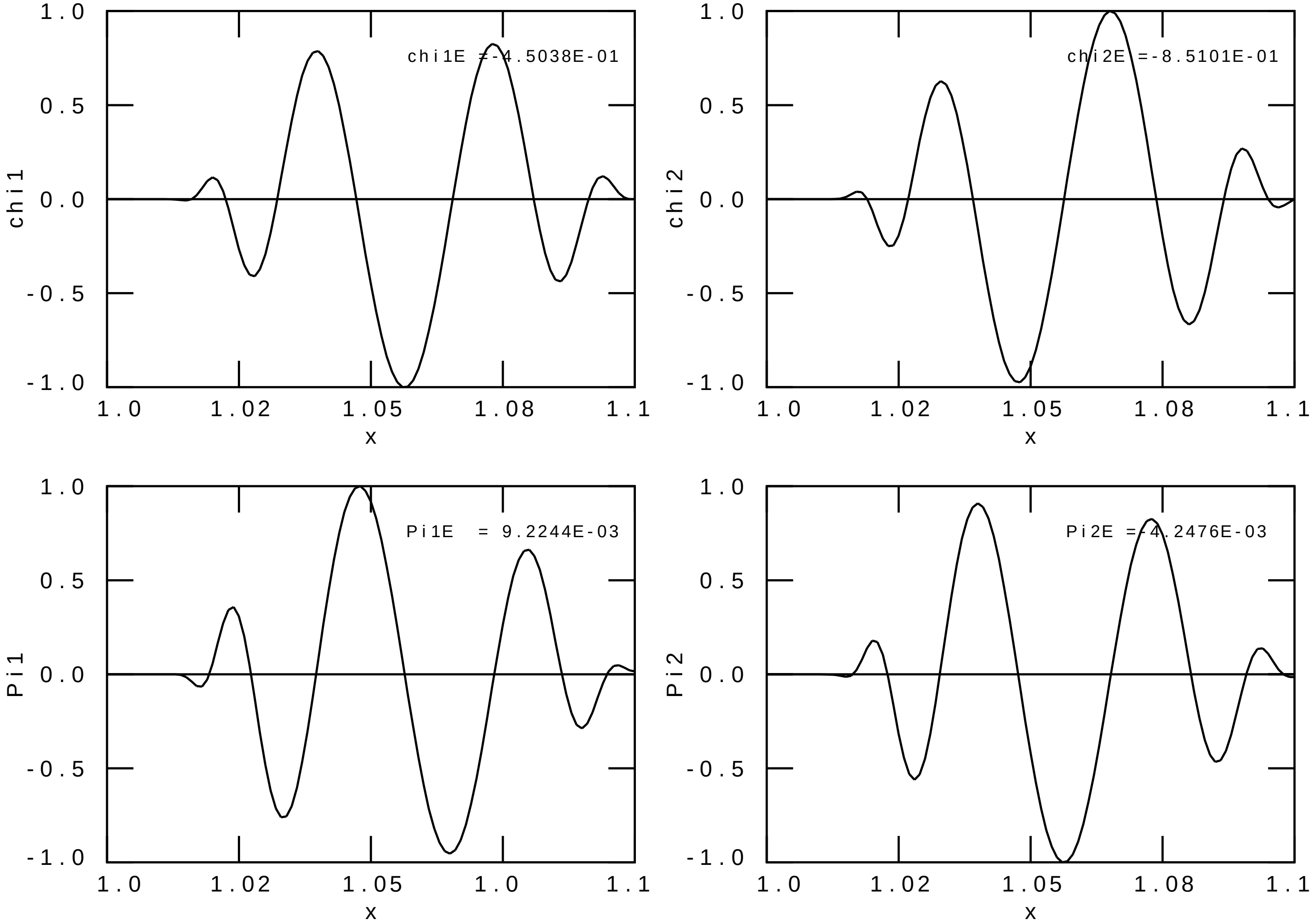
Spectral web (zoom)



Loops of solution path and alternator continue indefinitely towards the edges of forward and backward Alfvén/slow continua $\Omega^+_{\rm A,S}$ and $\Omega^+_{\rm A,S}$.

One of the cluster modes

$\sigma = -9.101\,,\ \nu = 0.06565$



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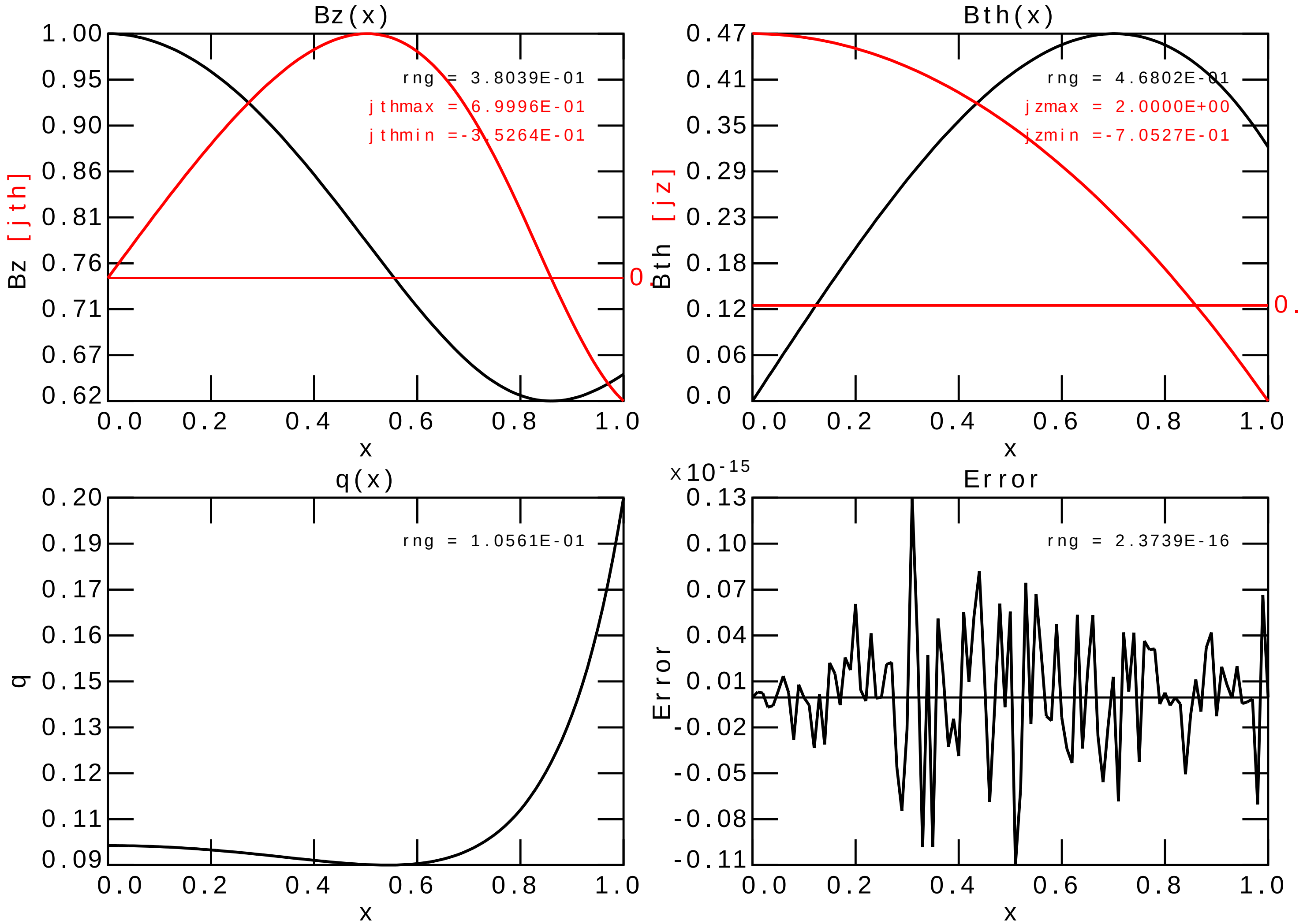
Equilibrium (ROC)

$\epsilon \equiv 2\pi a/L = 0.1\,,\ \ q_0 = 0.1\,,\ \ q_1 = 0.2$

K–S limit: $q_1 > 1$ (torus), $q_1 > 2$ (jet)



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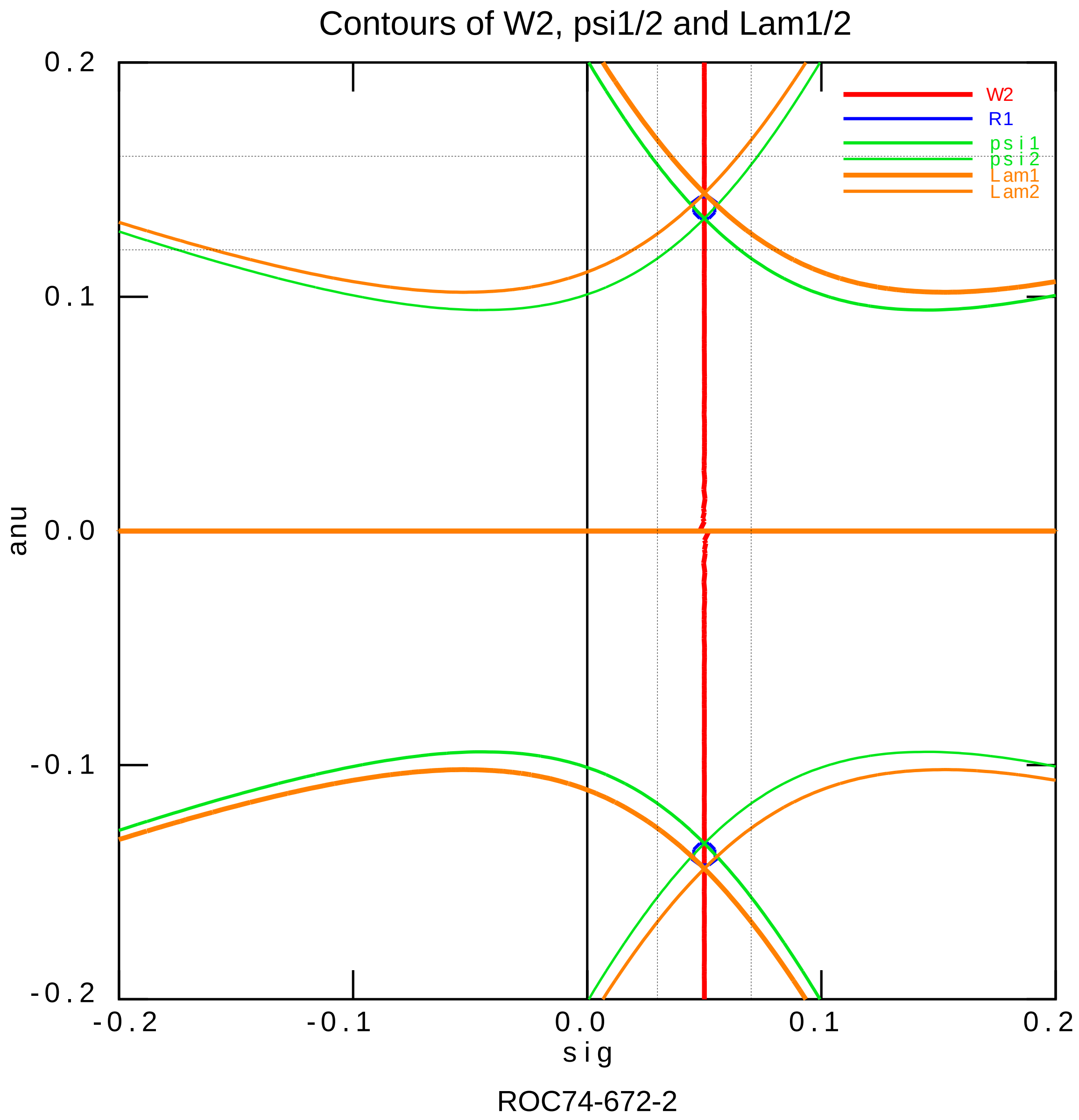
Spectral web (ROC)

$$k = n\pi/L\,,\ n = 1\,,\ m = -1$$

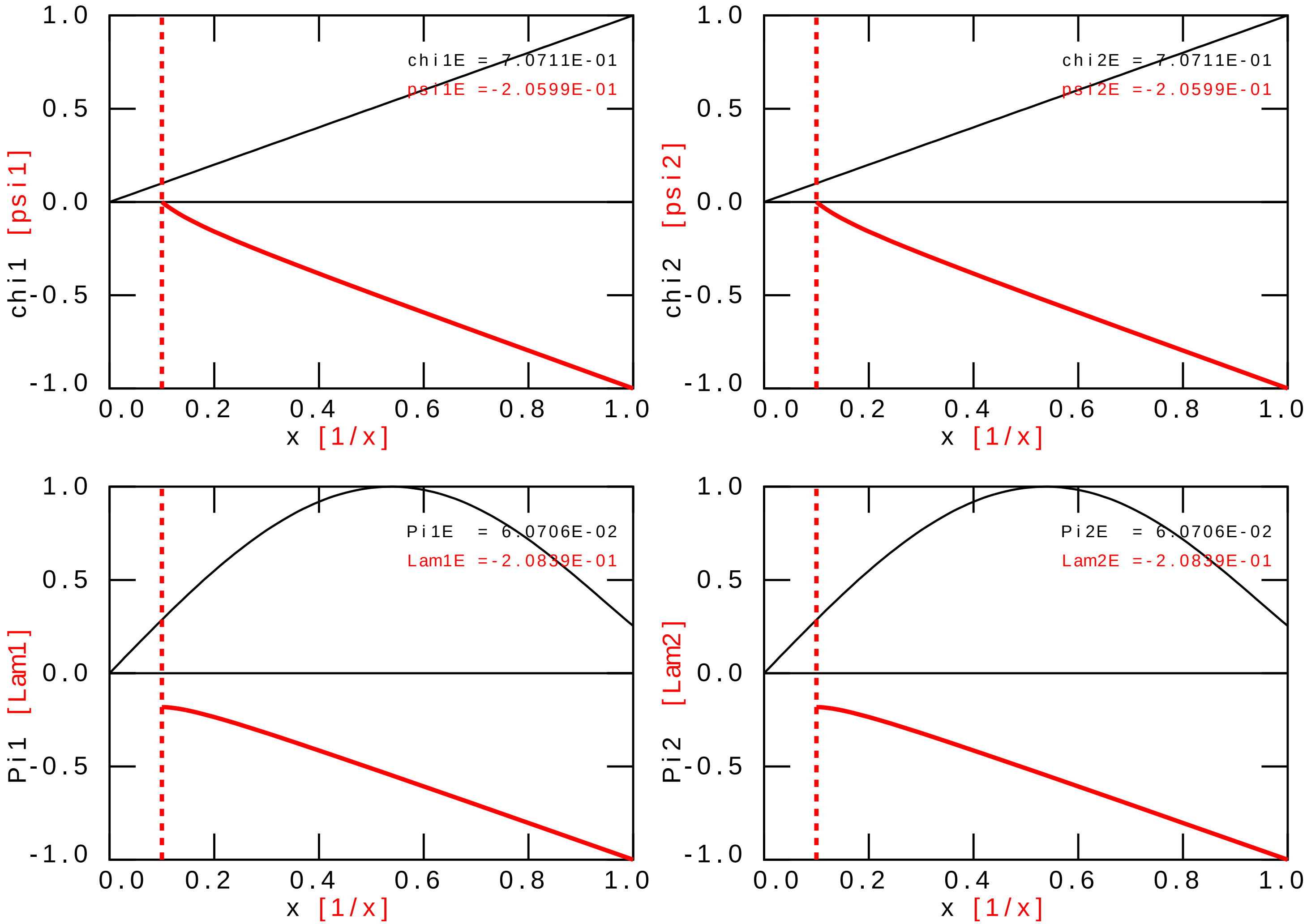
$\Downarrow$

Just one violently unstable external kink mode!



External kink mode

$$\sigma = -5.000 \times 10^{-2}\,,\ \nu = 0.1334$$



ROC74-672-3

Plasma:

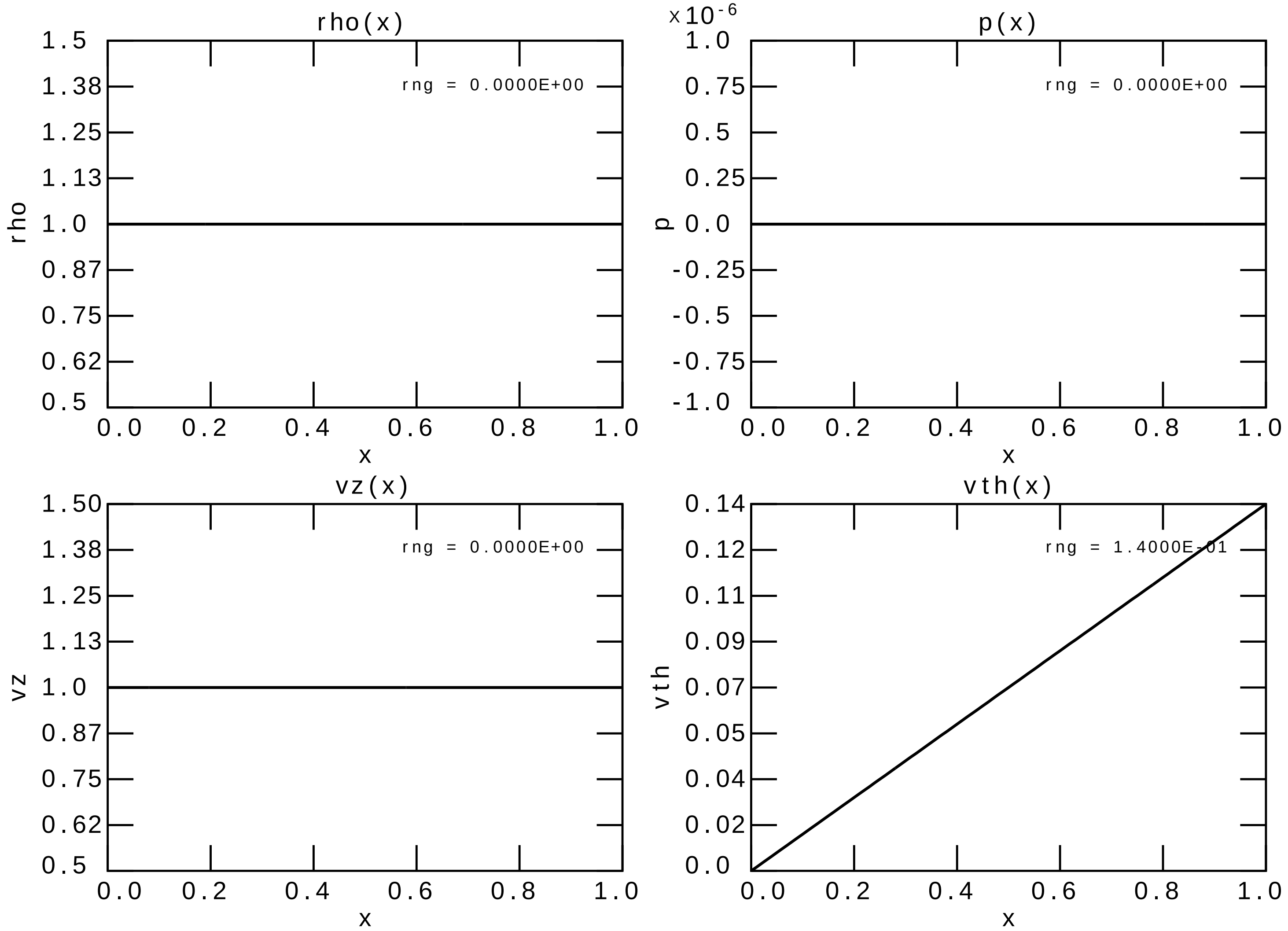
$(\chi\,,\Pi)^{\mathrm{T}}$,

'Vacuum':

$(\psi\,,\Lambda)^{\mathrm{T}}$.

Equilibrium (ROC)

Adding rigid rotation $(v_1 = 0.14)$



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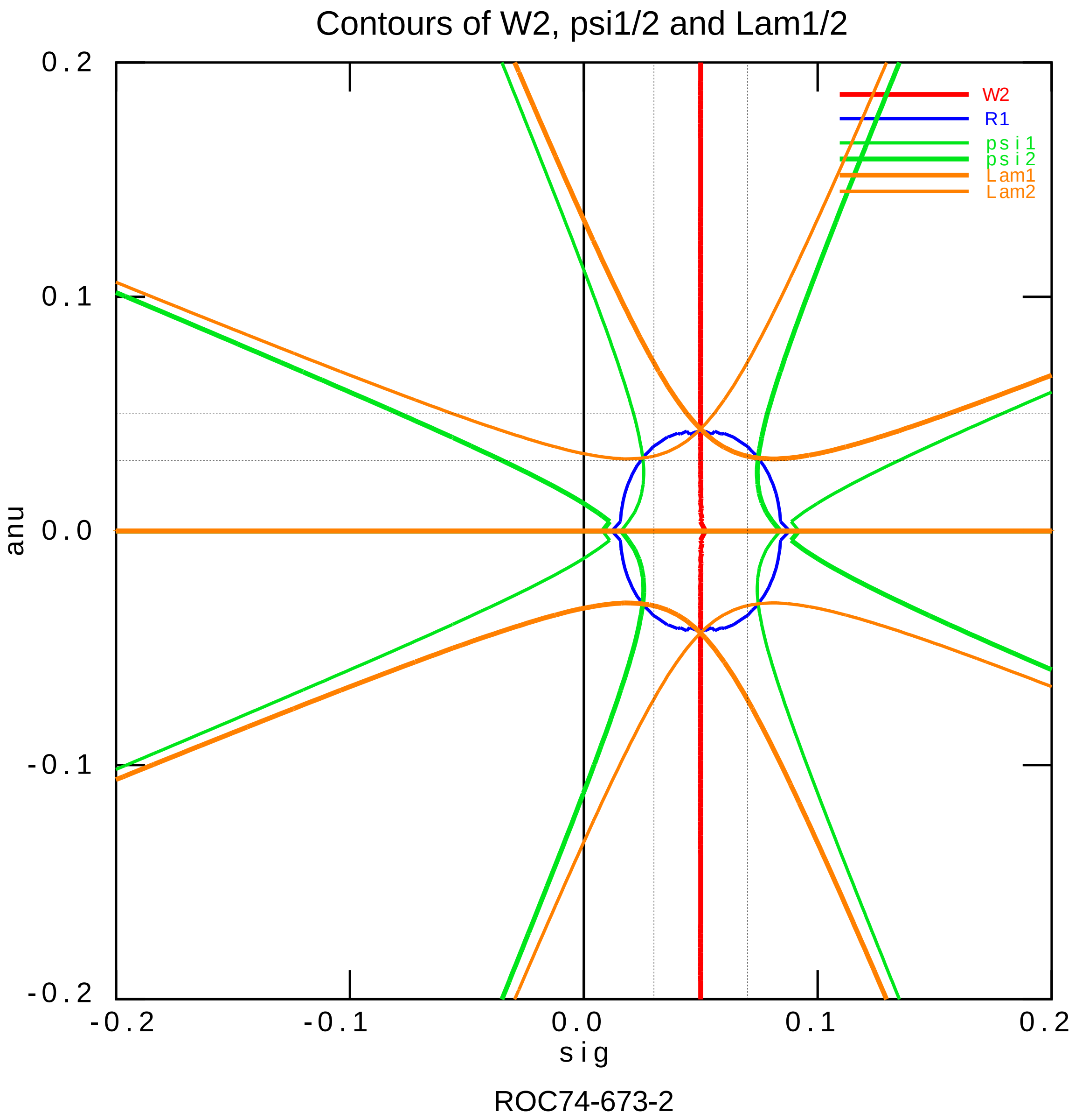
Spectral web (ROC)

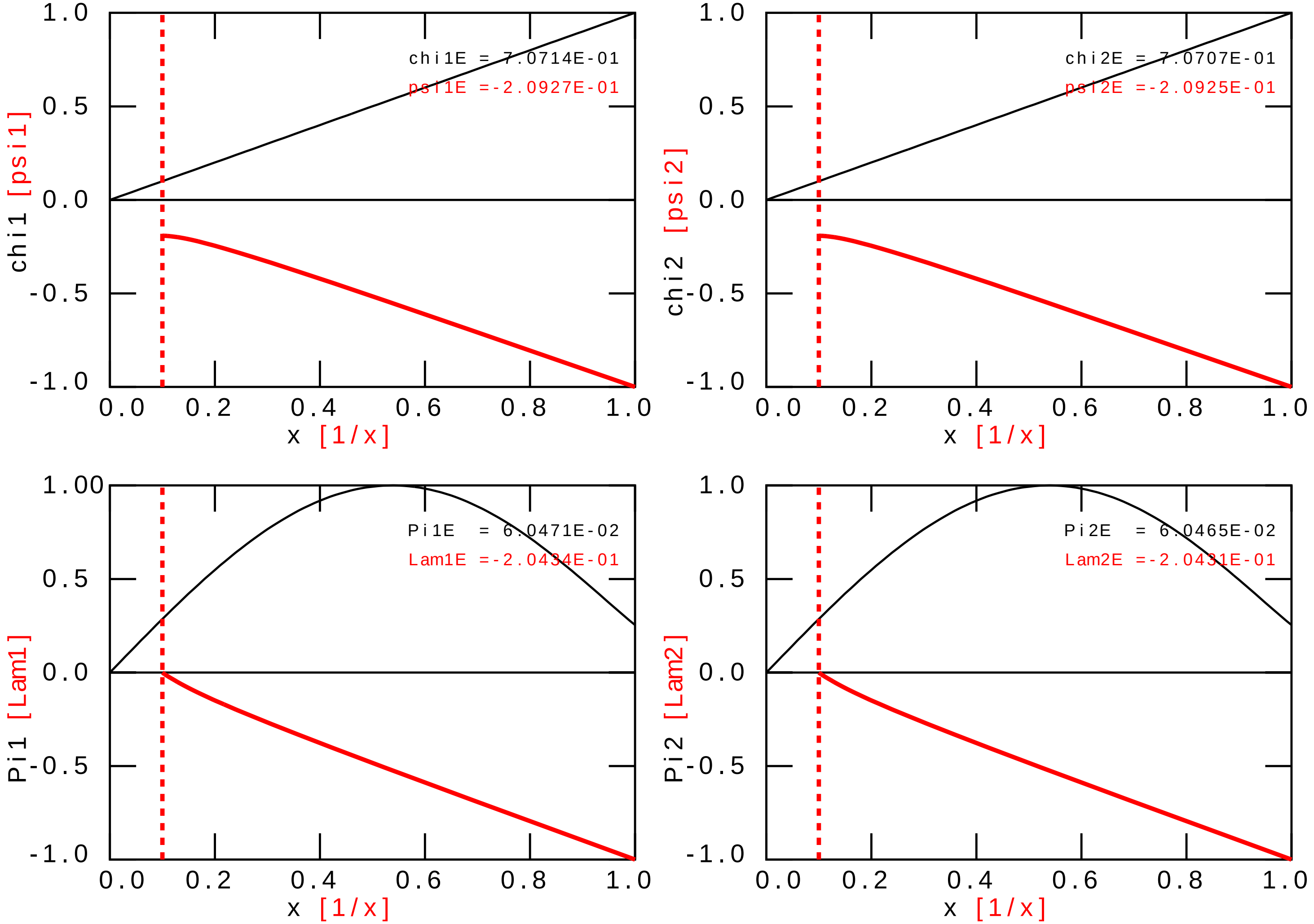


Approaching $v_1 = 0.15 \approx \frac{1}{3} v_{A,\theta}$ **where external kink mode is completely stabilized by rotation.**

External kink mode

$$\sigma = -4.989 \times 10^{-2}\,,\ \nu = 4.354 \times 10^{-2}$$

(nearly stable)



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Conclusions

New spectral theory

[Goedbloed, PoP (2009), PPCF (2011)]

- Construction of full complex MHD spectrum of moving plasmas based on self-adjointness of force operator $\mathbf{G}$ and Doppler–Coriolis operator U.

Method

- Closed system is opened up, converting the original EVP into one-sided BVP. Solvable for all complex ω, which makes the energy $\overline{W}$ complex, whereas the Doppler–Coriolis shift $\overline{V}$ remains real. $\Rightarrow$ $\boxed{W_2 = 0}$ provides the solution path, on which the alternator is real and monotonic and $\boxed{R_1 = 0}$ provides the EVs.

Applications

- Spectral web of MRIs and new class of non-axisymmetric modes.
- External kink modes of Alfvénic jets stabilized by rigid rotation.