MHD instabilities of accretion disks and jets
– a new spectral theory of rotating plasmas –

Hans Goedbloed & Rony Keppens

FOM-Institute for Plasma Physics ‘Rijnhuizen’
Astronomical Institute, Utrecht University
Center for Plasma Astrophysics, K.U. Leuven

[goedbloed@rijnh.nl]
1. Introduction
   • Fusion & astrophysical plasmas ⇒ Spectral theory of moving plasmas
   • Theme of new textbook on Advanced MHD ⇒ Stationary plasma flow

2. New spectral theory
   • Self-adjoint operators $G$ and $U$ ⇒ Real quadratic forms $W$ and $V$
   • Energy flow in open system ⇒ Solution path in the complex $\omega$ plane
   • Oscillation theorem ⇒ Alternator monotonic on solution path

3. Applications
   • Spectral web for magneto-rotational and other instabilities
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Two textbooks on Magnetohydrodynamics


Principles of Magnetohydrodynamics
With Applications to Laboratory and Astrophysical Plasmas

Hans Goedbloed
and Stefaan Poedts

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Advanced Magnetohydrodynamics
With Applications to Laboratory and Astrophysical Plasmas

J. P. Goedbloed
R. Keppens
and S. Poedts
1. Introduction:  The ideal MHD model (from Vol. 1)

Conservation laws & Scale independence

- **Ideal MHD equations** in terms of \( \rho, v, p, B \):

  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \quad \text{Conservation of mass}
  \]

  \[
  \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) + \nabla p - \rho g - \frac{1}{\mu_0} (\nabla \times B) \times B = 0, \quad \text{momentum}
  \]

  \[
  \frac{\partial p}{\partial t} + v \cdot \nabla p + \gamma p \nabla \cdot v = 0, \quad \text{entropy}
  \]

  \[
  \frac{\partial B}{\partial t} - \nabla \times (v \times B) = 0, \quad \nabla \cdot B = 0. \quad \text{magnetic flux!}
  \]

- **They are independent of length scale** \( (l_0) \), **density** \( (\rho_0) \) and **magnetic field** \( (B_0) \)

  \( \Rightarrow \) describe global dynamics of both laboratory and astrophysical plasmas!

- **Of course, to be supplemented with appropriate boundary conditions.**
Magnetized plasma is omni-present and described by magnetohydrodynamics.

- Tokamak (Iter)
- Pinwheel Galaxy M101 (HST)

⇒ **Nuts and bolts** fix static plasma
⇒ **Gravity and rotation** fix moving plasma
Fusion plasmas

- **Energy principle for static plasmas (1957):** standard stability paradigm for more than 50 years ⇒ interchanges, kinks, peeling–ballooning, RWM \((k \perp B)\).

- **Modification for stationary plasmas (Frieman–Rotenberg, 1960):** known, but hardly investigated due to misnomer “non self-adjoint”. Shear flow stabilizes some instabilities, but also drives new ones ⇒ Kelvin–Helmholtz (KH).

Astrophysical plasmas

- **Energy principle does not apply** since there are no static astrophysical plasmas.

- **Gravity and differential rotation establish equilibrium**, but also drive instabilities (violating tokamak “intuition”: \(k \parallel B\)) ⇒ Rayleigh–Taylor (RT), Parker, MRI, . . .

MHD spectroscopy of stationary plasma flow

- **Urgent common theme** for laboratory and astrophysical plasma research.

- **Demands fundamentally different approach** from static flow, that can be based on two foundations laid 50 (Frieman–Rotenberg) and 100 (Hilbert) years ago.
2. New spectral theory: Waves and instabilities of static plasmas

**Displacement vector**

⇒ Solves 3 of the 4 PDEs, so that only ‘Newton’s law’ remains:

\[ F(\xi) = \rho \frac{\partial^2 \xi}{\partial t^2} = -\rho \omega^2 \xi \]  
(for normal modes \( e^{i\omega t} \))

⇒ Energy:

\[ W \equiv -\frac{1}{2} \int \xi^* \cdot F(\xi) \, dV \]  
Hain *et al.* (1957), Bernstein *et al.* (1958)

⇒ More than 50 years applied to tokamaks, and even to astrophysical plasmas!
Modified displacement

Since astrophysical (and also present tokamak) plasmas are not static at all (even supersonic!)
⇒ Need MHD spectroscopy for moving plasmas.

⇒ Again solves 3 of the 4 PDEs so that ‘Newtons law’ remains:

\[
G(\xi) - 2\rho \mathbf{v} \cdot \nabla \frac{\partial \xi}{\partial t} - \rho \frac{\partial^2 \xi}{\partial t^2} = G(\xi) - 2\omega U \xi + \rho \omega^2 \xi = 0
\]

Frieman–Rotenberg (1960)

\( G \): generalized force operator, \( U \equiv -i\rho \mathbf{v} \cdot \nabla \): Doppler–Coriolis shift operator

(for plain waves \( e^{i\mathbf{k} \cdot \mathbf{x}} : = \mathbf{k} \cdot \mathbf{v}_0 \))
New spectral theory: Three fundamental problems

(1) \[ H\Psi = i\hbar \frac{\partial \Psi}{\partial t} = E\Psi \] (1926)

Quantum mechanics (atoms, molecules, condensed/living matter . . . everything?):
Hamiltonian \( H \Rightarrow \) real EVs \( E \rightarrow \) stable solutions!

(2) \[ F(\xi) = \rho \frac{\partial^2 \xi}{\partial t^2} = -\rho \omega^2 \xi \] (1957)

MHD of static plasmas (fusion only):
Force operator \( F \Rightarrow \) real EVs \( \omega^2 \rightarrow \) \( \begin{cases} W > 0 \ (\omega \text{ real}) & \rightarrow \text{stable waves} \\ W < 0 \ (\omega \text{ imag.}) & \rightarrow \text{instabilities} \end{cases} \)

(3) \[ G(\xi) - 2\rho v \cdot \nabla \frac{\partial \xi}{\partial t} - \rho \frac{\partial^2 \xi}{\partial t^2} = G(\xi) - 2\omega U \xi + \rho \omega^2 \xi = 0 \]

MHD of moving plasmas (fusion/astrophysical . . . cosmic):
Generalized force \( G \) and Doppler–Coriolis \( U \equiv -i\rho v \cdot \nabla \\
\Rightarrow EVs \begin{cases} \omega \text{ real} & \rightarrow \text{stable (undamped) waves} \\ \omega \text{ complex} & \rightarrow \text{instabilities/damped waves} \end{cases} \)

How to compute them?
Obstacle

- Problems (1) & (2): extensively studied (∼ ten thousands of papers).
- Problem (3): hardly studied (∼ hundreds of papers), due to widely held belief that “the problem is non-self-adjoint”.
- How come? Energy is conserved, and both $G$ and $U$ are self-adjoint!

Quadratic forms

- Inner product and norm: $\langle \xi, \eta \rangle \equiv \frac{1}{2} \int \rho \xi^* \cdot \eta \, dV$, $I[\xi] \equiv \|\xi\|^2 \equiv \langle \xi, \xi \rangle < \infty$.
- Operators are self-adjoint:

  $\langle \eta, \rho^{-1} U \xi \rangle = \langle \rho^{-1} U \eta, \xi \rangle \quad \Rightarrow \quad \text{real} \quad V \equiv \frac{1}{2} \int \xi^* \cdot U \xi \, dV$ (Doppler shift),

  $\langle \eta, \rho^{-1} G(\xi) \rangle = \langle \rho^{-1} G(\eta), \xi \rangle \quad \Rightarrow \quad \text{real} \quad W \equiv -\frac{1}{2} \int \xi^* \cdot G(\xi) \, dV$ (energy).

- But eigenvalue problem (3) is nonlinear:

  $\omega^2 - 2\overline{V} \omega - \overline{W} = 0$, \hspace{1cm} $\overline{V} \equiv V/I \equiv \langle \rho^{-1} U \rangle$, \hspace{1cm} $\overline{W} \equiv W/I \equiv \langle -\rho^{-1} G \rangle$. 
New spectral theory: Solution paths – monotonicity?

- ‘Solutions’ of the quadratic, with \( \omega \equiv \sigma + i\nu \):
  \[
  \begin{cases}
    \sigma = \overline{V} \pm \sqrt{W + V^2}, \quad \nu = 0 & \text{(stable waves)} \\
    \sigma = \overline{V}, \quad \nu = \pm \sqrt{-W - V^2} & \text{(instabilities)}
  \end{cases}
  \]
  
  This expression determines stability and yields a picture of where actual eigenvalues are located ⇒

- Would also yield a computational procedure if we knew:
  How to compute solution averages before eigenvalue (EV) is obtained?

- Recall static (‘linear’) eigenvalue problem:
  1. \( F \) self-adjoint ⇒ \( \omega^2 \) real ⇒ EVs \( \omega \) lie on the real and imaginary axes.
  2. EVs monotonic with number of zeros of \( \xi \) (Goedbloed–Sakanaka, 1974).

- In stationary problem, Doppler–Coriolis shift \( \overline{V} \) moves EVs off the imaginary axis:
  ⇒ (1) Solution path \( \equiv \) unknown curve on which the EVs are located?
  ⇒ (2) Monotonicity property of EVs on the solution path?
New spectral theory: Generic spectral problems

Arbitrary \( \rho(r), p(r), v_\theta(r), v_z(r), B_\theta(r), B_z(r) \),
but satisfying the equilibrium condition,
\[
(p + \frac{1}{2}B^2)' = (\rho v_\theta^2 - B_\theta^2)/r - \rho \Phi'_\text{gr}.
\]

Apply to two generic astrophysical problems:

1. **Accretion disk model, thin slice \( \Delta z \):**
   annulus \( \Delta r \) \((M_* \text{ at } r = 0), k \Delta z \gg 1; \)

2. **Rotating jet of finite length \( L \):**
   plasma + ‘vacuum’, \( \Phi_\text{gr} = 0, k = n\pi/L \).

Reduction of Frieman–Rotenberg equation, with \( \chi \equiv r \xi_r = \hat{\chi}(r)e^{i(m\theta + kz - \omega t)} \):

\[
\frac{d}{dr} \left[ \frac{N}{D} \frac{d\chi}{dr} \right] + \left[ A + \frac{B}{D} + \left\{ \frac{C}{D} \right\}' \right] \chi = 0, \\
\text{or} \\
N \frac{d}{dr} \begin{pmatrix} \chi \\ \Pi \end{pmatrix} + \begin{pmatrix} C & D \\ E & -C \end{pmatrix} \begin{pmatrix} \chi \\ \Pi \end{pmatrix} = 0,
\]

where \( N = N(r; \tilde{\omega}) \), with \( \tilde{\omega} \equiv \omega - k_0 \cdot v \), and \( \Pi \) is the total pressure perturbation.

**BCs:**

\[
\begin{align*}
\chi(r_1) &= 0 \quad \text{(left)} \\
\chi(r_2) &= 0 \quad \text{(right)}
\end{align*}
\]

\( \Rightarrow \) **Eigenvalue problem.**
Spectral properties

For plane slab: ODE similar to static case (Goedbloed, 1971), but $\omega$ is replaced by the **Doppler-shifted frequency** in co-moving layers:

$$\omega \rightarrow \tilde{\omega}(x) \equiv \omega - \Omega_0(x), \quad \Omega_0 \equiv k_0 \cdot v(x).$$

For cylinder: Hain–Lüst eq. (1958), generalized for rotation by Bondeson et al. (1987), and for gravitating thin disk with MRI by Keppens et al. (2002).

$$\Omega_0 = m v_\theta / r + k v_z,$$

**and Coriolis terms** $\sim v_\theta / r$ !!

Previous results:

- **HD** ⇒ **flow continuum** $\{\Omega_0(x)\}$, discovered by Case (1960).
- **MHD** ⇒ contrary to prolonged belief, **no flow continuum**! (Goedbloed *et al.*, 2004).
- Instead, three static MHD continua split into **six Doppler-shifted continua**:

$$\Omega_A^\pm \equiv \Omega_0(x) \pm \omega_A(x) \quad \text{(Alfvén)}, \quad \Omega_S^\pm \equiv \Omega_0(x) \pm \omega_S(x) \quad \text{(slow)}, \quad \Omega_F^\pm \equiv \pm \infty \quad \text{(fast)}.$$  

Flow continuum is obtained in the limit $B \rightarrow 0$.

⇒ How is the full (complex) spectrum connected to this (real) structure?
New spectral theory: Real EVs monotonic about the continua

Continuous spectra in HD and MHD

(Goedbloed, Beliën, van der Holst, Keppens, 2004)

**HD:**

(a) backward
p modes

(b) backward / forward

(g modes)

forward
p modes

**MHD:**

(b) backward

fast
Alfvén
slow

forward

slow
Alfvén
fast

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)

\( \Omega^+_p 0 \)

\( \Omega^-_p 0 \)

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)

\( \Omega^-_p \)

\( \Omega^0_p \)

\( \Omega^+_p \)
Consider open system

(a) **Spectral differential equation** can be solved accurately for arbitrary complex $\omega$.
   $\Rightarrow$ No problem!

(b) Actual problem is searching in the complex $\omega$-plane for the eigenvalues.
   $\Rightarrow$ Temporarily, drop that part by removing one of the boundaries!

Keep: $\xi(x_1) = 0$ (left BC)
Drop: $\xi(x_2) = 0$ (right BC).

- To get harmonic time dependence $\exp(-i\omega t)$, energy has to be injected or extracted at $x_2$.
  This is represented by imaginary part of energy, which we demand to vanish:
  
  $W_2 \equiv \text{Im}(W) = 0 \Rightarrow$ solution path!

- Required expression follows directly from proof of **self-adjointness of the force operator** $G$.

- Eigenvalues have to lie on this path.
Problem solvable for arbitrary complex $\omega$, but energy is complex: $W = W_1 + iW_2$.

$\Rightarrow$ Three BVPs:

I – Eigenvalues (closed),
wall on the plasma:
$W_2 = 0, \quad \xi(x_2) = 0$;

II – Solution path (closed), $\Leftarrow$
e.g. variable vacuum layer:
$W_2 = 0, \quad \xi(x_v) = 0$;

III – Arbitrary complex $\omega$ (open),
external excitation:
$W_2 \neq 0$.

Solution of the quadratic $\omega^2 - 2V\omega - \bar{W} = 0 \Rightarrow \sigma = \bar{V}$, iff $W_2 = 0$

$\Rightarrow$ average Doppler shifted real part of frequency vanishes iff system is closed.
Solution path

• **Pre-self-adjointness relation for** \( G \), with \( \xi \) and \( \eta \) not yet satisfying right BC:

\[
\int \left[ \eta^* \cdot G(\xi) - \xi \cdot G(\eta^*) \right] dV \overset{\text{(Gauss)}}{=} - \int \left[ \eta^* \Pi(\xi) - \xi \Pi(\eta^*) \right] dS \ (= 0 \text{ if BC}).
\]

Skip last step! Choosing \( \eta \equiv \xi^* \) yields easily computable expression for \( W_2 \):

\[
W_2 = \frac{1}{4} i \int \left[ \xi^* \cdot G(\xi) - \xi \cdot G(\xi^*) \right] dV = \frac{1}{2} \int (\xi_1 \Pi_2 - \xi_2 \Pi_1) dS
\]

\[
\Rightarrow W_2[\xi(r; \omega)] = 0 \Rightarrow \text{path } \mathcal{P}_u \text{ of unstable solutions}.
\]

• Equivalently, **self-adjointness of** \( U \) **yields mapping of** \( \omega \)-plane onto itself,

\[
Q(\omega) \equiv \omega - \overline{V}[\xi(r; \omega)] \equiv \omega - \frac{\int \xi^* \cdot U \xi dV}{\int \rho |\xi|^2 dV},
\]

which provides both solution paths:

\[
\begin{cases}
\text{Im } Q \equiv \nu = 0 \\
\text{Re } Q \equiv \overline{\sigma} \equiv \sigma - \overline{V}[\xi(r; \omega)] = 0
\end{cases} \Rightarrow \text{path } \mathcal{P}_s \text{ of stable solutions}, \quad \text{path } \mathcal{P}_u \text{ of unstable solutions}.
\]
Oscillation theorems and alternator

- Once solution path is determined, EVs on it are found by imposing the missing BC. But how does one move from one EV to the next?

- **Oscillation theorem \( \mathcal{R} \) for stable waves:** Counting nodes of the real function \( \xi \) yields Sturm–Liouville monotonicity (as static case: Goedbloed–Sakanaka 1974).

- **Instabilities:** On the solution path, the alternating ratio \( \mathcal{R} \equiv \frac{\xi}{\Pi} \) is real:

\[
W_2 = \frac{1}{2} [\xi_1 \Pi_2 - \xi_2 \Pi_1]_{x_2} = 0 \quad \Rightarrow \quad R \equiv \frac{\xi(x_2)}{\Pi(x_2)} = \frac{\xi_1(x_2)}{\Pi_1(x_2)} = \frac{\xi_2(x_2)}{\Pi_2(x_2)},
\]

\[
R_1 = 0 \quad \Rightarrow \quad \text{Eigenvalues}.
\]

- \( \Rightarrow \) **Oscillation theorem \( \mathcal{C} \) for instabilities** [proof exploits quadratic forms]:

The alternator \( \mathcal{R} \equiv \frac{\xi_e}{\Pi_e} \) is real and monotonic along the solution path in between the zeros of \( \Pi_e \) separating the eigenvalues.

- **Now, we are in business!**
3. Applications: (a) Rayleigh–Taylor & Kelvin–Helmholtz instabilities

Solution path

Plane gravitating slab

$\rho$: linear profile,

$\mathbf{B}$: sheared,

$\mathbf{v}$: sinusoidal profile.

$\Rightarrow$ Infinite sequence RT modes on ever smaller closed loops, one isolated KH mode.
**Full spectrum (LEDA–FLOW)**


Standard equilibrium:

\[ \rho = r^{-3/2}, \quad v_\theta \sim r^{-1/2}; \]

Parameters:

\[ \epsilon \equiv \sqrt{p} = 0.1, \]
\[ \beta \equiv 2p/B^2 = 2000, \]
\[ \mu \equiv B_\theta/B_z = 1; \]

Mode numbers:

\[ m = 0, \quad k = 50 \]
Applications: MRIs

Spectral web (ROC)

Equilibrium:
\[ \epsilon = 0.1, \quad \beta = 100, \quad \mu = 1; \]
Mode numbers:
\[ m = 0, \quad k = 50. \]

Solution path is not along imaginary axis:
\[ \sigma = \overline{V} \neq 0 \]

Alternator loops give genuine \((\xi_1 = \xi_2 = 0)\)
& false \((\Pi_1 = \Pi_2 = 0)\) eigenvalues.
Applications: MRI, eigenfunction nr. 1

Fastest growing mode

\[ \sigma = -2.031 \times 10^{-3}, \quad \nu = 0.6277 \]
Applications: MRI, eigenfunction nr.10

One of the cluster modes

\[ \sigma = -1.287 \times 10^{-3}, \, \nu = 0.3861 \]
Applications: (c) Non-axisymmetric modes (NAM)

Spectral web (ROC)

Contours of solution path and alternator

Equilibrium:
\[ \epsilon = 0.1, \quad \beta = 100, \quad \mu = 1; \]

Mode numbers:
\[ m = 10, \quad k = 50 \]

Overlapping continua:
\[ \Omega_{A,S}^+, \Omega_{A,S}^- \]

Both, solution path and alternator form loops!
Applications: NAM, eigenfunction nr. 1

Fastest growing mode

\[ \sigma = -8.860 \times 10^{-3}, \, \nu = 0.3753 \]
Loops of solution path and alternator continue indefinitely towards the edges of forward and backward Alfvén/slow continua \( \Omega_{A,S}^+ \) and \( \Omega_{A,S}^+ \).
One of the cluster modes

\[ \sigma = -9.101, \ \nu = 0.06565 \]
Applications: (d) Alfvénic jet (far beyond Kruskal–Shafranov limit!)

Equilibrium (ROC)

\[ \epsilon \equiv \frac{2\pi a}{L} = 0.1, \quad q_0 = 0.1, \quad q_1 = 0.2 \]

K–S limit: \( q_1 > 1 \) (torus), \( q_1 > 2 \) (jet)
Applications: Alfvénic jet

Spectral web (ROC)

$\kappa = n\pi/L$, $n = 1$, $m = -1$

Just one violently unstable external kink mode!
Applications: Alfvénic jet

**External kink mode**

\[ \sigma = -5.000 \times 10^{-2}, \nu = 0.1334 \]

Plasma:
\[ (\chi, \Pi)^T \]

‘Vacuum’:
\[ (\psi, \Lambda)^T. \]
Applications: Rotating Alfvénic jet

Equilibrium (ROC)

Adding rigid rotation \( (v_1 = 0.14) \)
Approaching $v_1 = 0.15 \approx \frac{1}{3}v_{A,\theta}$ where external kink mode is completely stabilized by rotation.
Applications: Rotating Alfvénic jet

External kink mode

\[ \sigma = -4.989 \times 10^{-2}, \ \nu = 4.354 \times 10^{-2} \]
(nearly stable)

\( \sigma \) and \( \nu \) are constants in the context of magnetic configurations, indicating the stability of the kink mode.
Conclusions

New spectral theory  
[Goedbloed, PoP (2009), PPCF (2011)]

- Construction of full complex MHD spectrum of moving plasmas based on self-adjointness of force operator $G$ and Doppler–Coriolis operator $U$.

Method

- Closed system is opened up, converting the original EVP into one-sided BVP. Solvable for all complex $\omega$, which makes the energy $\overline{W}$ complex, whereas the Doppler–Coriolis shift $\overline{V}$ remains real. ⇒ $W^2 = 0$ provides the solution path, on which the alternator is real and monotonic and $R_1 = 0$ provides the EVs.

Applications

- Spectral web of MRIs and new class of non-axisymmetric modes.
- External kink modes of Alfvénic jets stabilized by rigid rotation.