Diffusive cosmic ray acceleration at relativistic shock waves with magnetostatic turbulence

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Diffusive cosmic ray acceleration at relativistic shock waves with magnetostatic turbulence; RS, 2015, ApJ, submitted (arXiv: :1503.04737)

Cosmic-ray transport in accelerating flows; J. G. Kirk, P. Schneider and RS, 1988, ApJ 328, 269

1. Introduction

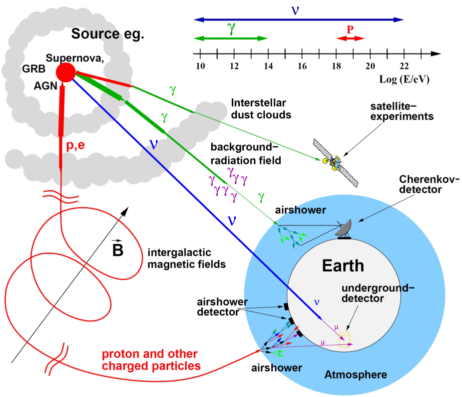

Figure 1: Sketch of cosmic ray life. Courtesy R. Wagner.

Fig. 1 sketches the typical life of a cosmic ray particle: after being accelerated in individual sources such like supernova remnants, pulsar wind nebulae, active galactic nuclei or gamma-ray bursts, it stochastically propagates in the partially turbulent magnetic field and interacts with the ambient photon and matter fields, generating nonthermal photon and neutrino radiation.

Diffusive first-order Fermi acceleration at nonrelativistic shock fronts has been regarded as a prime candidate for particle acceleration in astrophysics (for reviews see Drury 1983; Blandford and Eichler 1987). Modern TeV air-Cherenkov telescopes have indeed resolved the shock regions in supernova remnants and identified the shocks as strong emission regions of TeV photons generated by the accelerated particles (Hinton and Hofmann 2009).

Pulsar wind nebulae, active galactic nuclei and gamma-ray bursts exhibit highly collimated winds or jets with initial bulk Lorentz factors $\Gamma_0 = (1-(V_0/c)^2)^{-1/2}$ up to $100 - 10^3$. Magnetized shock waves with relativistic speeds form due to the interaction of the relativistic supersonic and super-Alfvenic outflows with the ambient ionized interstellar or intergalactic medium. There is high interest to understand the acceleration of cosmic rays at magnetized relativistic shocks.

While for nonrelativistic shock waves the analytic theory of diffusive shock acceleration is well developed (Axford et al. 1977, Krymsky 1987, Blandford and Ostriker 1978, Bell 1978), for relativistic shock speeds such an analytical theory did not exist until recently even for parallel shock waves, although the underlying Fokker-Planck transport equation (see Eq. (3) below) for the particle dynamics has already been derived (Webb 1985; Kirk, Schneider and RS 1988) some years ago.

The existing literature concentrated on semi-numerical eigenfunction solutions of this Fokker-Planck transport equation pioneered by Kirk and Schneider (1987), relativistic Monte Carlo simulations (Ellison et al. 1990, Ostrowski 1991, Bednarz and Ostrowski 1998, Summerlin and Baring 2012), and relativistic particle in-cell simulations (Spitkovsky 2008, Sironi and Spitkovsky 2009).

Here I describe my recent attempt to develop an analytical study of cosmic ray acceleration in parallel relativistic magnetized shock waves employing the diffusion approximation in the upstream and downstream regions of the shock wave. The development runs much in parallel with the existing work on nonrelativistic shocks.

2. Particle transport equations

Magnetized space plasmas harbour low-frequency linear ($\delta B \ll B_0$) transverse MHD waves (such as shear Alfven and magnetosonic plasma waves) with phase and group speeds $V_{\rm ph} = V_g \leq V_A$, where the Alfven speed $V_A = 2.18 \cdot 10^{11} B(\mathsf{G}) n_e^{-1/2}(\mathsf{cm}^{-3})$ cm s^{-1} is highly subluminal ($\ll c$) *in the rest frame of the moving plasma*. Faraday$'$s induction law then indicates for MHD waves that the strength of turbulent electric fields $\delta E = (V_A/c)\delta B \ll \delta B$ is much smaller than the strength of turbulent magnetic fields.

The ordering $B_0 \gg \delta B \gg \delta E$ corresponds to the derivation of cosmic ray transport equations for $< f > (\vec{X}, p, \mu, \phi, t) \to f_0(\vec{X}, p, \mu, t) \to F(\vec{X}, p, t)$ from the collisionfree Boltzmann equation for the full phase space distribution $< f > (\vec{X}, p, \mu, \phi, t)$ to the Fokker-Planck equation for its gyrotropic part $f_0(\vec{X}, p, \mu, t)$, and to the diffusion-convection transport equation for its isotropic part $F(\vec{X}, p, t)$, respectively. Accordingly, the cosmic ray anisotropy, defined as the deviation

$$g(\vec{X}, p, \mu, t) = f_0(\vec{X}, p, \mu, t) - F(\vec{X}, p, t), \tag{1}$$

then is small ($|g| \ll F$) with respect to F. The diffusion approximation applied to the Fokker-Planck transport equation for $f_0(\vec{X}, p, \mu, t)$ allows us to relate the cosmic ray anisotropy g to the solutions of the diffusion-convection tranport equation for F.

$$\vec{X} = (X, Y, Z) = \vec{x} + \frac{c}{q_a B_0^2} \vec{p} \times \vec{B}_0 = \vec{x} + \frac{c}{q_a B_0} \begin{pmatrix} p_y \\ -p_x \\ 0 \end{pmatrix}, \qquad (2)$$

denote the coordinates of the cosmic ray guiding center, where we orient the large-scale guide magnetic field, which is uniform on the scales of the cosmic ray particles gyradii $R_L = v/|\Omega|$, $\vec{B}_0 = B_0 \vec{e}_z = (0, 0, B_0)$ along the z-axis. $v = \beta c$ and $\Omega_a = q_a B_0 / \gamma m_a c$ denote the speed and the relativistic gyrofrequency of a cosmic ray particle with mass m_a, charge q_a and energy $\gamma m_a c^2$.

The Larmor-phase averaged Fokker-Planck transport equation in a medium with magnetostatic turbulence, propagating with the stationary bulk speed $\vec{U}$ with $\Gamma = [1 - (U^2/c^2)]^{-1/2}$ aligned along the magnetic field direction, is given by (Webb 1985; Kirk, Schneider and RS 1988)

$$\Gamma \left[1 + \frac{U v \mu}{c^2}\right] \frac{\partial f_0}{\partial t^*} + \Gamma \left[U + v\mu\right] \frac{\partial f_0}{\partial z^*} + \frac{v(1-\mu^2)}{2L} \frac{\partial f_0}{\partial \mu} + \mathcal{R} f_0 - S(\vec{X}^*, p, t^*)$$

$$- \alpha(z^*) \left(\mu + \frac{U}{v}\right) \left[\mu p \frac{\partial f_0}{\partial p} + (1-\mu^2) \frac{\partial f_0}{\partial \mu}\right] = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial f_0}{\partial \mu}\right] \qquad (3)$$

irrespective of how the pitch-angle Fokker-Planck scattering coefficient $D_{\mu\mu}(\mu)$ is calculated, either by quasilinear (RS 2002) or nonlinear (Shalchi 2009) cosmic ray transport theories.

$$\alpha(z^*) = \frac{c^2}{U(z^*)} \frac{d\Gamma(z^*)}{dz^*} = \frac{dU}{dz^*}\Gamma^3 = \frac{d(U\Gamma)}{dz^*} \tag{4}$$

denotes the rate of adiabatic deceleration/acceleration in relativistic flows. The focusing length (Roelof 1969)

$$L = -\frac{B_0(z^*)}{(dB_0(z^*)/dz^*)} \tag{5}$$

accounts for a possible large-scale spatial gradient of the guide magnetic field B_0, and

$$\mathcal{R}f = -p^{-2}\frac{\partial}{\partial p}\left[p^2 \dot{p}_{\rm loss} f\right] + \frac{f}{T_c} \tag{6}$$

represents continuous ($\dot{p}_{\rm loss}$) and catastrophic (T_c) momentum losses of cosmic ray particles.

Most importantly: (1) in the Fokker-Planck Eq. (3) the phase space coordinates have to be taken in the mixed comoving coordinate system: time t^* and space coordinates z^* in the laboratory (=observer) system and particle's momentum coordinates p and $\mu = p_\parallel/p$ in the rest frame of the streaming plasma.

(2) The flow dependent last terms on the left-hand side of Eq. (3),

$$T = \alpha(z^*) \left(\mu + \frac{U}{v}\right) \left[\mu p \frac{\partial f_0}{\partial p} + (1-\mu^2) \frac{\partial f_0}{\partial \mu}\right], \tag{7}$$

plays an important role. It has to be treated correctly both for nonrelativistic and relativistic shock speeds.

It vanishes for spatially constant flows as the rate of adiabatic deceleration/acceleration (4) vanishes. Then the remaining flow velocity (U) dependent terms in Eq. (3) simply result from the Lorentz transformation of special relativity of the rest-frame position-time coordinates (z, t) to the laboratory-frame position-time coordinates (z^*, t^*).

However, for spatially varying flow speeds $U(z^*)$ special relativity no longer applies and has to be replaced by the transformation laws from general relativity. As noted by Riffert (1986) as well as Kirk, Schneider and Schlickeiser (1988) these introduce connection coefficients or Christoffel symbols of the first kind. In a flat Euclidean space-time metric the terms (7) are exactly these connection coefficients.

2.1. Diffusion approximation

Dropping from now on the (z^*, t^*)-notation, we split the total density f_0 as in Eq. (1) into the isotropic part F and an anisotropic part g,

$$f_0(\vec{X}, p, \mu, t) = F(\vec{X}, p, t) + g(\vec{X}, p, \mu, t), \quad \int_{-1}^{1} d\mu \, g(\vec{X}, p, \mu, t) = 0 \qquad (8)$$

Inserting the ansatz (8) in the Fokker-Planck equation (3) and averaging over μ yields the diffusion-convection transport equation

$$\Gamma \left[\frac{\partial F}{\partial t} + U \frac{\partial F}{\partial z}\right] - \frac{\alpha}{3} p \frac{\partial F}{\partial p} + \mathcal{R} F + \frac{v}{2}\left[\Gamma\left(\frac{\partial}{\partial z} + \frac{U}{c^2}\frac{\partial}{\partial t}\right) + \frac{1}{L}\right] \int_{-1}^{1} d\mu \, \mu g$$
$$- \frac{\alpha U}{2v}\left[p\frac{\partial}{\partial p} + 2\right] \int_{-1}^{1} d\mu \, \mu g - \frac{\alpha}{2}\left[p \frac{\partial}{\partial p} + 3\right] \int_{-1}^{1} d\mu \, \mu^2 g = S(\vec{X}, p, t), \qquad (9)$$

involving the first two moments of the anisotropy. The identities for any function $Y(p)$

$$\left[p\frac{\partial}{\partial p} + 3\right] Y(p) = \frac{1}{p^2}\frac{\partial}{\partial p} p^3 Y(p) \qquad (10)$$

and

$$\left[\frac{p}{v}\frac{\partial}{\partial p}+\frac{2}{v}\right]Y(p)=p\frac{\partial}{\partial p}\left(\frac{Y}{v}\right)+\frac{3-\beta^2}{v}Y=\frac{1}{p^2}\frac{\partial}{\partial p}\left(\frac{p^3Y(p)}{v}\right)-\frac{\beta^2Y}{v},\quad (11)$$

where $\beta=v/c$, provide

$$\frac{\alpha U}{2}\left[\frac{p}{v}\frac{\partial}{\partial p}+\frac{2}{v}\right]Y(p)=\frac{\alpha U}{2p^2}\frac{\partial}{\partial p}\left(\frac{p^3Y(p)}{v}\right)-\frac{\alpha U\beta^2Y}{2v}$$

$$=\frac{\alpha}{2p^2}\frac{\partial}{\partial p}\left(\frac{p^3UY(p)}{v}\right)-\frac{v}{2}\frac{d\Gamma}{dz}Y,\quad (12)$$

where we inserted $\alpha(z)$ from Eq. (4).

The diffusion-convection transport equation (9) then reads

$$\Gamma\left[\frac{\partial F}{\partial t}+U\frac{\partial F}{\partial z}\right]-\frac{\alpha}{3}p\frac{\partial F}{\partial p}+\mathcal{R}F+\frac{v}{2}\left[\left(\frac{\partial}{\partial z}\Gamma+\frac{\Gamma U}{c^2}\frac{\partial}{\partial t}\right)+\frac{1}{L}\right]\int_{-1}^{1}d\mu\,\mu g$$

$$-\frac{\alpha}{2p^2}\frac{\partial}{\partial p}p^3\left[\int_{-1}^{1}d\mu\,\mu^2g+\frac{U}{v}\int_{-1}^{1}d\mu\,\mu g\right]=S(\vec{X},p,t)\quad (13)$$

Subtracting Eq. (13) from the Fokker-Planck equation (3) provides the equation for the anisotropy

$$
\begin{aligned}
&\Gamma v\mu\left(\frac{\partial F}{\partial z}+\frac{U}{c^2}\frac{\partial F}{\partial t}\right)+\Gamma v\mu\left(\frac{\partial g}{\partial z}+\frac{U}{c^2}\frac{\partial g}{\partial t}\right)+\Gamma\left(\frac{\partial g}{\partial t}+U\frac{\partial g}{\partial z}\right)\\
&+\mathcal{R}g-\alpha\left(\mu+\frac{U}{v}\right)\left[\mu p\frac{\partial F}{\partial p}+\mu p\frac{\partial g}{\partial p}+(1-\mu^2)\frac{\partial g}{\partial \mu}\right]\\
&+\frac{\alpha}{3}p\frac{\partial F}{\partial p}-\frac{v}{2}\left[\left(\frac{\partial}{\partial z}\Gamma+\frac{\Gamma U}{c^2}\frac{\partial}{\partial t}\right)+\frac{1}{L}\right]\int_{-1}^{1}d\mu\,\mu g\\
&+\frac{\alpha}{2p^2}\frac{\partial}{\partial p}p^3\left[\int_{-1}^{1}d\mu\,\mu^2 g+\frac{U}{v}\int_{-1}^{1}d\mu\,\mu g\right]\\
&+\frac{v(1-\mu^2)}{2L}\frac{\partial g}{\partial \mu}=\frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial g}{\partial \mu}\right]
\end{aligned}
\tag{14}
$$

We note that Eqs. (13) and (14) are still exact.

2.2. Approximated anisotropy and anisotropy moments

We chose as laboratory frame the rest frame of the shock wave with the step-wise velocity profile

$$U(z) = \begin{cases} -U_1 = -\beta_1 c = \text{const.} & \text{for } 0 < z \le \infty \text{ (upstream)} \\ -U_2 = -\beta_2 c = \text{const.} & \text{for } -\infty \le z < 0 \text{ (downstream)} \end{cases} \tag{15}$$

In this case the rate of adiabatic acceleration (4)

$$\alpha = \alpha_0 \delta(z), \;\; \alpha_0 = -(U_1\Gamma_1 - U_2\Gamma_2) \tag{16}$$

is non-zero only at the position of the shock.

With $|g| \ll F$ we approximate the anisotropy equation (14) to leading order by

$$\Gamma v \mu \frac{\partial F}{\partial z} \simeq \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial g}{\partial \mu}\right], \tag{17}$$

where we also neglect the time derivate of F as compared to the spatial gradient of F, i.e. $U/c^2(\partial F/\partial t) \ll (\partial F/\partial z)$.

Integrating Eq. (17) twice provides for the cosmic ray anisotropy

$$g(\vec{X},p,\mu,t) \simeq \frac{\Gamma v}{4}\Big[\int_{-1}^{1} d\mu\, \frac{(1-\mu)(1-\mu^2)}{D_{\mu\mu}(\mu)} - 2\int_{-1}^{\mu} dx\, \frac{(1-x^2)}{D_{\mu\mu}(x)}\Big]\frac{\partial F}{\partial z} \qquad (18)$$

to determine the two moments needed in the diffusion-convection transport equation (13) as

$$\int_{-1}^{1} d\mu\, \mu g = -\frac{\Gamma v K_0}{4}\frac{\partial F}{\partial z}, \quad \int_{-1}^{1} d\mu\, \mu^2 g = -\frac{\Gamma v K_1}{6}\frac{\partial F}{\partial z} \qquad (19)$$

in terms of the two $(n=0,1)$ integrals

$$K_n = \int_{-1}^{1} d\mu\, \frac{\mu^n(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \qquad (20)$$

For the often considered case of symmetric pitch-angle Fokker-Planck coefficients $D_{\mu\mu}(-\mu) = D_{\mu\mu}(\mu)$ the integral $K_1 = 0$ vanishes.

3. Diffusion-convection transport equation for relativistic flows

With these moments the diffusion-convection transport equation (13) becomes after straightforward algebra

$$\Gamma\frac{\partial F}{\partial t}-\frac{\kappa_{zz}\Gamma}{L}\frac{\partial F}{\partial z}+\frac{\partial}{\partial z}\left[\Gamma\left(UF-\Gamma\kappa_{zz}\frac{\partial F}{\partial z}\right)\right]+\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\kappa_{pz}\Gamma\frac{\partial F}{\partial z}-\frac{\alpha p^3 F}{3}\right]$$
$$+\mathcal{R}F=S(\vec{X},p,t) \tag{21}$$

with the two diffusion coefficients

$$\kappa_{zz}=\frac{v^2K_0}{8}=\frac{v^2}{8}\int_{-1}^{1}d\mu\,\frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)},\ \ \kappa_{pz}=\frac{\alpha vp}{12}\left(K_1+\frac{3U}{2v}K_0\right)$$
$$=\frac{\alpha vp}{12}\left(\int_{-1}^{1}d\mu\,\frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)}+\frac{3U}{2v}\int_{-1}^{1}d\mu\,\frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}\right) \tag{22}$$

Eq. (21) is the diffusion-convection transport equation of cosmic rays in aligned parallel flows of arbitrary speed containing magnetostatic slab turbulence with the cosmic ray phase space coordinates taken in the mixed comoving coordinate system. It is particularly appropriate to investigate cosmic ray particle acceleration in parallel relativistic flows.

3.1. Nonrelativistic limit

In the limit of nonrelativistic flows $U(z) \ll c$ so that $\Gamma \simeq 1.$ the transport equation (21) becomes

$$\frac{\partial F}{\partial t} - \frac{\kappa_{zz}}{L}\frac{\partial F}{\partial z} + \frac{\partial}{\partial z}\left[UF - \kappa_{zz}\frac{\partial F}{\partial z}\right] + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\kappa_{pz}\frac{\partial F}{\partial z} - \frac{\alpha p^3 F}{3}\right] + \mathcal{R}F = S(\vec{X}, p, t) \tag{23}$$

Eq. (23) with $L = \infty$ differs from the transport theory used in earlier nonrelativistic diffusive shock acceleration theory (Axford et al. 1977, Krymsky 1987, Blandford and Ostriker 1978, Bell 1978, Drury 1983) by the additional third last term on the left-hand side involving κ_{pz}, which results from our correct handling of the connection coefficients (7).

As we will demonstrate below, this additional term provides a major modification of the resulting differential momentum spectrum of accelerated particles in the nonrelativistic flow limit at nonrelativistic particles momenta: instead of a power law distribution of accelerated particles at the shock a Lorentzian distribution function results, which at large momenta then approaches the power law distribution inferred in earlier acceleration theories for nonrelativistic shock speeds.

3.2. Magnetostatic slab Alfven waves

There are four different magnetostatic slab Alven waves: forward and backward propagating waves which each can be left- or right-handed circularly polarized, respectively. In terms of the cross helicity H_c and the magnetic helicities $\sigma_\pm$ of forward and backward moving Alfven waves, and power-law type wave intensities $I \propto k_\parallel^{-s}$ with $s \in (1,2)$ with the same spectral index s of all four waves (isospectral turbulence), the two integrals (20) are given by

$$K_n = \frac{64}{s-1} \frac{(R_L k_{\min})^{2-s}}{v k_{\min}} \left(\frac{B_0}{\delta B}\right)^2 S_n(H_c, \sigma_+, \sigma_-) \tag{24}$$

for $R_L k_{\min} \leq 1$ with

$$S_0 = \frac{2}{(2-s)(4-s)G(H_c, \sigma_+, \sigma_-)}, \quad S_1 = \frac{Z\left[\sigma_+ + \sigma_- + H_c(\sigma_+ - \sigma_-)\right]}{(3-s)(5-s)G(H_c, \sigma_+, \sigma_-)}, \tag{25}$$

where

$$G(H_c, \sigma_+, \sigma_-) = 4 - \left[\sigma_+ + \sigma_- + H_c(\sigma_+ - \sigma_-)\right]^2 \tag{26}$$

is always positive. $Z = q/\mathrm{q}$ the sign of the cosmic ray particle and $k_{\min}$ the smallest wavenumber of the Alfven waves with total magnetic field strength $(\delta B)^2$.

Both integrals (24) exhibit the same momentum dependence $K_n \propto (p/|q|)^{2-s} v^{-1}$.
Moreover, S_1 depends on the cosmic ray charge sign.

With the maximum cosmic ray momentum with $q = Qe$

$$p_m = \frac{|Q|eB_0}{k_{\rm min}c} = 1.5\cdot 10^{14}|Q|B(\mu\mathrm{G})\lambda_1 \ \ \frac{\mathrm{eV}}{c}, \tag{27}$$

where $\lambda_{\rm max} = 2\pi k_{\rm min}^{-1} = 1\lambda_1$ pc denotes the maximum wavelength of the Alfven waves, we obtain for the integrals (24) $K_n(p > p_m) = 0$ and

$$K_n(p \le p_m) = K\frac{S_n}{v}\left(\frac{p}{p_m}\right)^{2-s} \tag{28}$$

with the constant length

$$K = \frac{32}{\pi(s-1)}\left(\frac{B_0}{\delta B}\right)^2\lambda_{\rm max} = 3.1\cdot 10^{19}\lambda_1\left(\frac{B_0}{\delta B}\right)^2 \ \mathrm{cm} \tag{29}$$

Consequently, we find for the diffusion coefficients (22)

$$\kappa_{zz} = \frac{vK}{8}S_0\left(\frac{p}{p_m}\right)^{2-s}, \ \ \kappa_{pz} = \alpha K_{pz},$$

$$K_{pz} = \frac{pK}{12}\left(S_1 + \frac{3US_0}{2v}\right)\left(\frac{p}{p_m}\right)^{2-s} \tag{30}$$

4. Particle acceleration at relativistic shock waves

We adopt the step-like shock profile (15) and assume particle injection $S(\vec{X},p,t) = S(p)\delta(z)$ at the position of the shock only with the injection momentum spectrum $S(p)$. Moreover, we assume spatially constant flow velocities and diffusion coefficients in the upstream and downstream region.

In the steady-state case with no losses ($\mathcal{RF} = 0$) and a uniform background magnetic field ($L = \infty$) the diffusion-convection transport equation (21) in the rest frame of the shock wave reduces to

$$\frac{\partial}{\partial z}\left[\Gamma\left(UF - \Gamma\kappa_{zz}\frac{\partial F}{\partial z}\right)\right] + \frac{\alpha_0\delta(z)}{p^2}\frac{\partial}{\partial p}\left[p^2 K_{pz}\Gamma\frac{\partial F}{\partial z} - \frac{p^3 F}{3}\right] = S(p)\delta(z) \tag{31}$$

We solve Eq. (31) by the same method as for nonrelativistic parallel step-like shock waves. For the upstream ($z > 0$) and downstream ($z < 0$) regions

$$F_1(z>0,p) = F_0(p)\exp\left[-\frac{U_1 z}{\Gamma_1\kappa_{zz,1}}\right], \quad F_2(z<0,p) = F_0(p), \tag{32}$$

which approach zero far upstream $z \to \infty$ and are finite far downstream $z \to -\infty$. At the position of the shock

$$F_1(z=0,p) = F_2(z=0,p) = F_0(p) \tag{33}$$

the distribution function is continuous.

4.1. Momentum spectrum at the shock

The particle momentum spectrum $F_0(p)$ at the position of the shock is obtained by integrating the transport equation (31) from $z=-\eta$ to $z=\eta$ and considering the limit $\eta \to 0$. This provides the continuity condition for the cosmic ray streaming density at the shock

$$-\Gamma_1\left(U_1F_1+\Gamma_1\kappa_{zz,1}\frac{\partial F_1}{\partial z}\right)_{0^+}+\Gamma_2\left(U_2F_2+\Gamma_2\kappa_{zz,2}\frac{\partial F_2}{\partial z}\right)_{0^-}$$
$$+\frac{\alpha_0}{p^2}\frac{\partial}{\partial p}\left[p^2K_{pz,1}\Gamma_1\left(\frac{\partial F_1}{\partial z}\right)_{0^+}-\frac{p^3F_0}{3}\right]=S(p) \qquad (34)$$

With the up- and downstream solutions (32) we obtain

$$\Gamma_2U_2F_0(p)+\frac{U_1\Gamma_1-U_2\Gamma_2}{3p^2}\frac{d}{dp}\left(p^3+\frac{3p^2K_{pz,1}U_1}{\kappa_{zz,1}}\right)F_0(p)=S(p) \qquad (35)$$

According to Eqs. (30) we find for the case of slab Alfven waves for the ratio

$$\frac{3K_{pz,1}U_1}{\kappa_{zz,1}}=p\frac{U_1}{v}\left[\frac{2S_1}{S_0}+\frac{3U_1}{v}\right]=p\frac{U_1}{v}\left[ZR(s,\sigma_+,\sigma_-,H_c)+\frac{3U_1}{v}\right] \qquad (36)$$

with the helicity dependent function

$$R(s,\sigma_+,\sigma_-,H_c)=\frac{(2-s)(4-s)}{(3-s)(5-s)}\left[\sigma_++\sigma_-+H_c(\sigma_+-\sigma_-)\right] \qquad (37)$$

The value of the bracket of this function is not greater than 2 for all helicity values, so the function is limited to values

$$|R_{\max}| \leq \frac{2(2-s)(4-s)}{(3-s)(5-s)}, \tag{38}$$

which for all values of $s \in [1, 2)$ is smaller than 0.75.
In the case of symmetric pitch-angle Fokker-Planck coefficients $D_{\mu\mu}(-\mu) = D_{\mu\mu}(\mu)$ so that $K_1 = 0$ the ratio (36) simplifies to

$$\frac{3K_{pz,1}U_1}{\kappa_{zz,1}} = 3p\frac{U_1^2}{v^2} \tag{39}$$

For both cases Eq. (35) then reads

$$F_0(p) + \frac{1}{\psi p^2}\frac{d}{dp}\left(p^2 T(p) F_0(p)\right) = \frac{S(p)}{\Gamma_2 U_2} \tag{40}$$

where we define the ratio

$$\psi = \frac{3U_2\Gamma_2}{U_1\Gamma_1 - U_2\Gamma_2} = \frac{3}{\sqrt{\frac{r^2-\beta_1^2}{1-\beta_1^2}} - 1} \tag{41}$$

in terms of the shock wave compression ratio $r = U_1/U_2 = \beta_1/\beta_2$, and the function

$$T(p) = p\left(1 + \frac{mc\beta_1\sqrt{1+(p/mc)^2}}{p}\left[ZR + \frac{3mc\beta_1\sqrt{1+(p/mc)^2}}{p}\right]\right) \quad (42)$$

For a monomomentum injection spectrum $S(p) = S_0\delta(p-p_0)$ we obtain

$$F_0(p \geq p_0) = \frac{3S_0}{U_1\Gamma_1 - U_2\Gamma_2}\frac{p_0^2}{p^2 T(p)}e^{-\psi I(p,p_0)} \quad (43)$$

with the integral

$$I(p,p_0) = \int_{p_0}^{p}\frac{dx}{T(x)} = \int_{\frac{mc}{p}}^{\frac{mc}{p_0}}\frac{dy}{y\left[1+\beta_1\sqrt{1+y^2}[ZR+3\beta_1\sqrt{1+y^2}]\right]} \quad (44)$$

For the differential number density of accelerated particles $N(p) = 4\pi p^2 F(p)$ the solution (43) implies

$$N_0(p \geq p_0) = N_2(z<0,p) = 4\pi p^2 F(p \geq p_0) = \frac{4\pi S_0\psi}{U_2\Gamma_2}\frac{p_0^2}{T(p)}e^{-\psi I(p,p_0)},$$

$$N_1(z>0,p) = N_0(p \geq p_0)\exp\left[-\frac{U_1 z}{\Gamma_1\kappa_{zz,1}}\right] \quad (45)$$

5. Symmetric pitch-angle Fokker-Planck coefficient

For symmetric pitch-angle Fokker-Planck coefficients ($R = 0$) the integral (44) reduces to

$$I(R=0) = \int_{p_0/mc}^{p/mc} \frac{dy}{y\left[1+3\beta_1(1+\frac{1}{y^2})\right]} = \frac{1}{2(1+3\beta_1^2)} \ln \frac{1+\frac{p^2}{p_c^2}}{1+\frac{p_0^2}{p_c^2}} \tag{46}$$

with the characteristic momentum

$$p_c(\beta_1) = \sqrt{\frac{3\beta_1^2}{1+3\beta_1^2}} mc \simeq \begin{cases} \sqrt{3} m U_1 & \text{for } \beta_1 \ll 1 \\ mc & \text{for } \Gamma_1 \gg 1 \end{cases} \tag{47}$$

Then the differential number density at the shock (45) becomes the Lorentzian-type distribution function

$$N_0(p \geq p_0) = A_0 p \left[1 + (\frac{p}{p_c})^2\right]^{-\rho}, \;\; A_0 = \frac{4\pi S_0 \psi p_0^2}{U_2 \Gamma_2 (1+3\beta_1^2) p_c^2} \left[1 + \frac{p_0^2}{p_c^2}\right]^{\rho - 1} \tag{48}$$

illustrated in Fig. 2, with

$$\rho = \frac{\psi}{2(1+3\beta_1^2)} + 1 \tag{49}$$

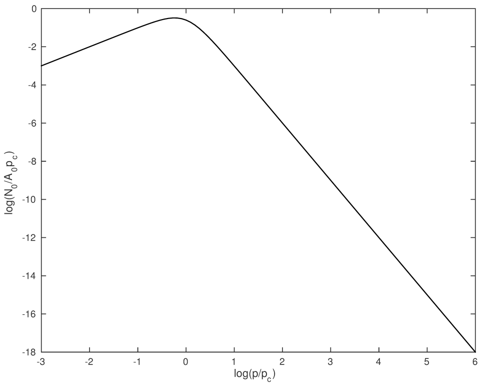

Figure 2: Differential number density (48) of accelerated particles at the shock as a function of p/p_c in the case $R = 0$ for the adopted spectral index value $\rho = 2$ and injection momentum $p_0/p_c = 10^{-3}$.

For particle momenta $p_0 \leq p \leq p_c$ the Lorentzian distribution (48) increases linearly with momentum, $N_0(p_0 \leq p \leq p_c) \simeq A_0 p$, whereas for large momenta $p \geq p_c$ it approaches the decreasing power law distribution

$$N_0(p \geq p_c) \simeq A_0 p_c \left(\frac{p}{p_c}\right)^{-\xi} \tag{50}$$

with the power law spectral index

$$\xi = 2\rho - 1 = 1 + \frac{\psi}{1 + 3\beta_1^2} = 1 + \frac{3}{(\Gamma_1 \sqrt{r^2 - \beta_1^2} - 1)(1 + 3\beta_1^2)} \tag{51}$$

5.1. Nonrelativistic shock waves

For nonrelativistic shock velocities $\beta_1 \ll 1$, so that $\Gamma_1 \simeq 1$, Eq. (49) becomes

$$\rho \simeq \frac{\psi_0}{2} + 1 = \frac{2r+1}{2(r-1)}, \tag{52}$$

In this case the Lorentzian distribution function (48) reads

$$N_0(p \geq p_0) = A_0 p \left[1 + (\frac{p}{\sqrt{3}\beta_1 mc})^2\right]^{-\frac{2r+1}{2(r-1)}}, \tag{53}$$

approaching at momenta $p > p_c^{\rm nr} = \sqrt{3}\beta_1 mc$ the decreasing power law distribution

$$N_0(p \geq p_c^{\rm nr}) \simeq A_0 p_c^{\rm nr} \left(\frac{p}{p_c}\right)^{-\xi_0}, \quad \xi_0 = \frac{r+2}{r-1} \tag{54}$$

This spectral index agrees with the standard result for nonrelativistic shocks providing $\xi_0 \geq 2$ for shocks in adiabatic electron-proton media with compression ratios $r \leq 4$ and $\xi_0 \geq 3/2$ for shocks in adiabatic electron-positron media with compression ratios $r \leq 7$.

5.2. Relativistic shock waves

For relativistic shock velocities with $\beta_1 \simeq 1$ and $\Gamma_1 \gg 1$, Eq. (49) becomes

$$\rho \simeq 1 + \frac{\psi}{8} \simeq 1 + \frac{3}{8(\Gamma_1\sqrt{r^2-1}-1)}, \tag{55}$$

but now we have to distinguish between particle injection at nonrelativistic ($p_0 \ll mc$) and at relativistic ($p_0 \gg mc$) momenta. In the first case

$$N_0(p \geq p_0) = A_0 p \left[1 + (\frac{p}{mc})^2\right]^{-1-\frac{3}{8(\Gamma_1\sqrt{r^2-1}-1)}}, \tag{56}$$

approaching at relativistic particle momenta

$$N_0(p \geq mc) \simeq A_0 mc \left(\frac{p}{mc}\right)^{-\xi(\Gamma_1 \gg 1)}, \;\; \xi(\Gamma_1 \gg 1) = 1 + \frac{3}{4(\Gamma_1\sqrt{r^2-1}-1} \tag{57}$$

For $\Gamma_1 \gg 1$ the spectral index is close to unity.

If cosmic rays are injected at relativistic momenta $p_0 \gg mc$ the power law limit (50) of the distribution function (48) holds, so that with $p_c \simeq mc$ again Eq. (57) results. As $p_0 \gg p_c$ we obtain

$$N_0(p \geq p_0 \gg mc) \simeq \frac{4\pi S_0 \psi p_0}{U_2\Gamma_2(1+3\beta_1^2)} \left(\frac{p}{p_0}\right)^{-\xi(\Gamma_1 \gg 1)} \tag{58}$$

6. Relativistic cosmic rays

For relativistic particle momenta the integral (44) can be solved for general values of the helicity dependent function R. We assume here that cosmic ray particles are injected at relativistic momenta $p_0 \gg mc$ with the monomomentum injection spectrum $S(p) = S_0\delta(p - p_0)$. With $p \geq p_0 \gg mc$ the integral (44) reduces to

$$I(p \geq p_0 \gg mc) \simeq \frac{1}{1 + \beta_1 ZR + 3\beta_1^2} \ln\left(\frac{p}{p_0}\right) \tag{59}$$

Consequently, the solution (45) becomes the power law distribution

$$N_0(p \geq p_0 \gg mc) \simeq \frac{4\pi S_0 p_0 \psi}{U_2\Gamma_2(1 + \beta_1 ZR + 3\beta_1^2)} \left(\frac{p}{p_0}\right)^{-\xi} \tag{60}$$

with the power law spectral index

$$\xi = 1 + \frac{\psi}{1 + \beta_1 ZR + 3\beta_1^2} = 1 + \frac{3}{\left(\Gamma_1\sqrt{r^2 - \beta_1^2} - 1\right)(1 + \beta_1 ZR + 3\beta_1^2)} \tag{61}$$

We first note that for $R = 0$ the power law solution (60) agrees with the earlier derived expression (58) and that the spectral indices (61) and (51) agree.

6.1. Nonrelativistic shock waves

For nonrelativistic shock velocities $U_{1,2} \ll c$, so that $\Gamma_1 \simeq 1 + (\beta_1^2/2)$ and $\Gamma_2 \simeq 1 + (\beta_1^2/2r^2)$, the particle power law spectral index (61) to first order in $\beta_1 \ll 1$ reduces to

$$\xi = 1 + \frac{3[1 - ZR\beta_1]}{r-1} = \frac{r + 2 - 3ZR\beta_1}{r-1} \tag{62}$$

To lowest order in β_1 we again reproduce the standard result for nonrelativistic shocks $\xi(\beta_1 = 0) = (r+2)/(r-1)$.

However, our result (62) gives a small (for turbulence spectral indices $s < 2$) correction to this standard spectral index which is different for positively ($Z = 1$) and negatively ($Z = -1$) charged cosmic ray particles. Depending on the sign of helicity dependent function R defined Eq. (36) this implies either a smaller or greater spectral index compared to the standard result. With the maximum value (38) the correction is at most

$$|\Delta\xi| \le \frac{3R_{\max}\beta_1}{r-1} = \frac{6(2-s)(4-s)}{(3-s)(5-s)}\frac{\beta_1}{r-1}, \tag{63}$$

which for a Kolmogorov turbulence spectral index $s = 5/3$ gives

$$|\Delta\xi| \le \frac{3R_{\max}\beta_1}{r-1} = 1.05\frac{\beta_1}{r-1}, \tag{64}$$

For most adiabatic shocks with $r > 1.1$ this is negligibly small as $|\Delta\xi| \le 10.5\beta_1$.

6.2. Relativistic shock waves

The determination of the power law spectral indices (61) and (51) require the knowledge of the shock compression ratio $r = \beta_1/\beta_2$ which for relativistic shocks depends for any given shock speed β_1 in a non-trivial way on the equations of state of the up- and downstream fluids as shown for hyrodynamical shocks by Peacock (1981), Heavens and Drury (1988) and Kirk and Duffy (1999).

The jump conditions for relativistic magnetohydrodynamic shocks in gyrotropic plasmas were studied by Double et al. (2004) and Gerbig and RS (2011), including the pressure anisotropy $\chi = P_\perp/P_\parallel$ of the upstream and downstream gas pressures adopting adiabatic equation of states of the up- and down-stream gas with adiabatic indices $\kappa_{1,2}$.

For illustrating our results we consider here only the case of an ultrarelativistic shock $\Gamma_1 \gg 1$ and a relativistic downstream medium with adiabatic index $4/3$, so that $\beta_1 = 3\beta_2$ (Blandford and McKee 1976) or $r = 3$. In this case we obtain for the power law spectral indices (61) and (51)

$$\xi(\Gamma_1 \gg 1) = 1 + \frac{3}{\left[\sqrt{8\Gamma_1^2+1}-1\right]\left[4-\frac{3}{\Gamma_1^2}+ZR\sqrt{1-\frac{1}{\Gamma_1^2}}\right]}$$

$$\simeq 1 + \frac{3}{2\sqrt{2}[4+ZR]\Gamma_1} \qquad (65)$$

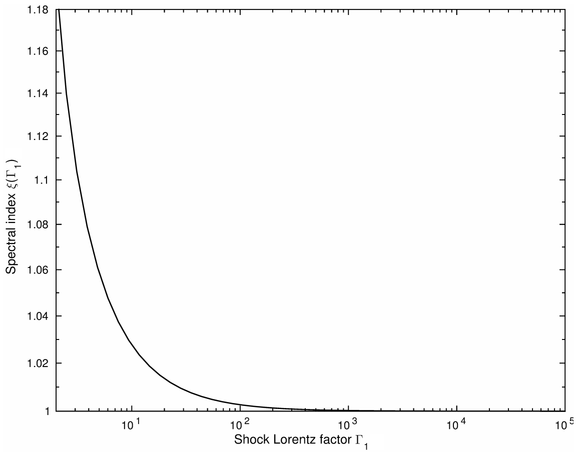

Figure 3: Power law spectral index (65) of relativistic particles accelerated at an ultrarelativistic shock for the case $R = 0$ as a function of the shock Lorentz factor Γ_1.

In Fig. 3 we calculate this spectral index for the case $R = 0$ for relativistic shocks with $\Gamma_1 \geq 2$, indicating spectral index values close to unity. Due to the dominating Γ^{-1} dependence of $\xi - 1$ the limit $\xi \simeq 1$ is reached for $\Gamma_1 > 10$. This proves that ultrarelativistic shocks accelerate relativistic cosmic rays (for negligible losses) very efficiently with power law spectral indices close to unity.

Our result of flat spectral indices with $\xi \simeq 1$ for ultrarelativistic shocks disagrees strongly with the earlier established universal spectral index value $\xi \in [2.25 - 2.30]$ from the eigenfunction and Monte Carlo simulation studies (for review see Kirk and Duffy 1999).

As possible explanation for this difference we recall that our analytical solution is based on the two continuity conditions (33) and (34) at the shock. These two continuity conditions are needed as our steady-state diffusion-convection transport equation (31) is a second-order differential equation in the position coordinate z.

While the continuity condition (33) for the particle phase density at the shock is also used in the eigenfunction solution method, the continuity condition (34) for the flux of particles is not used in that method as the Fokker-Planck transport equation (3) is a first-order differential equation in the position coordinate z. It is clear that the use of different continuity conditions results in different results.

7. Summary and conclusions

- The analytical theory of diffusive cosmic ray acceleration at parallel stationary shock waves with magnetostatic turbulence is generalized to arbitrary shock speeds $V_s = \beta_1 c$, including in particular relativistic speeds. This is achieved by applying the diffusion approximation to the relevant Fokker-Planck particle transport equation formulated in the mixed comoving coordinate system.

- The Fokker-Planck particle transport equation contains connection coefficients resulting from the coordinate transformations into this mixed frame which are properly included in the diffusion approximation. For magnetostatic slab turbulence the diffusion-convection transport equation for the isotropic (in the rest frame of the streaming plasma) part of the particle's phase space density is derived for the first time for arbitrary shock speeds.

- In the limit of nonrelativistic flows the diffusion-convection transport equation differs from the transport equation used in earlier nonrelativistic diffusive shock acceleration theory by an additional term, resulting from the correct handling of the connection coefficients. The additional term implies a modification of the resulting differential momentum spectrum of accelerated particles in the nonrelativistic flow limit at nonrelativistic particles momenta: a Lorentzian distribution function results, which at large momenta then approaches the power law distribution inferred in earlier acceleration theories for nonrelativistic shock speeds.

- For a step-wise shock velocity profile the steady-state diffusion-convection transport equation is solved analytically for the first time for arbitrary shock speeds, following closely the solution method developed for nonrelativistic speeds, making use of the continuity conditions for the cosmic ray phase space density and streaming density at the shock.

- For a symmetric pitch-angle scattering Fokker-Planck coefficient $D_{\mu\mu}(-\mu) = D_{\mu\mu}(\mu)$ the steady-state solution is independent of the microphysical scattering details.

- For nonrelativistic mono-momentum particle injection at the shock the differential number density of accelerated particles is a Lorentzian-type distribution function which at large momenta approaches a power law distribution function $N(p \geq p_c) \propto p^{-\xi}$ with the spectral index $\xi(\beta_1) = 1 + [3/(\Gamma_1\sqrt{r^2 - \beta_1^2} - 1)(1 + 3\beta_1^2)]$.

- For nonrelativistic $(\beta_1 \ll 1)$ shock speeds this spectral index agrees with the known result $\xi(\beta_1 \ll 1) \simeq (r+2)/(r-1)$, whereas for ultrarelativistic $(\Gamma_1 \gg 1)$ shock speeds the spectral index value is close to unity. If particle injection occurs already at relativistic momenta, the steady-state solution is of power law type at all higher particle momenta.

- For asymmetric pitch-angle scattering Fokker-Planck coefficient $D_{\mu\mu}(-\mu) \neq D_{\mu\mu}(\mu)$, resulting from magnetostatic isospectral slab Alfven waves with non-zero values of the magnetic and cross helicities, the momentum spectrum of accelerated particles depends on the microphysical details of particle's pitch angle scattering. In particular, a dependence of the momentum spectrum on the charge sign of the cosmic ray particles is found.

- Ultrarelativistic shocks accelerate relativistic cosmic rays (for negligible losses) very efficiently with power law spectral indices close to unity.

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