A Dynamo with a Global Helicity constraint for a Galaxy with a Corona

Avijeet Prasad (avijeet@iiap.res.in)
Arun Mangalam (mangalam@iiap.res.in)

Indian Institute of Astrophysics, Bangalore
October 20, 2014

MOTIVATION

- Helicity conservation imposes constraints on dynamo action and one can study the minimal field strength of the large scale magnetic field that could arise as a result.
- As a first step, we have analytically build a galactic disk dynamo model in presence of a force-free corona and study its steady-state solutions.
- We develop a formalism for obtaining time-dependent solutions that are expressed in terms of these steady-state solutions.

AXISYMMETRIC DYNAMO EQUATION

Assumptions

1. Axisymmetry in **U** and **B**
2. α is independent of time
3. $\alpha - \omega$ dynamo is operative

The dynamo equation :

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{U}} \times \bar{\mathbf{B}}) + \alpha \bar{\mathbf{B}} + (\eta + \eta_T) \nabla^2 \bar{\mathbf{B}}. \quad (1)$$

Express **B** as a combination of poloidal flux ψ and poloidal current function T as (Mangalam & Subramanian 1994)

$$\mathbf{B_P} = B_r \hat{r} + B_z \hat{z} = \frac{1}{r} \nabla \psi \times \hat{\phi} = \hat{\mathbf{P}} \psi \quad (2)$$

$$\mathbf{B}_\phi = \frac{T}{r} \hat{\phi} \quad (3)$$

DYNAMO EQUATIONS

The dynamo equation expressed in terms of ψ and T now reduces to

$$\left(\frac{\partial}{\partial t} - \eta\Lambda\right)\psi = \alpha T \tag{4}$$

$$\left(\frac{\partial}{\partial t} - \eta\Lambda\right)T = -r\frac{\mathrm{d}\omega}{\mathrm{d}r}\frac{\partial\psi}{\partial z} \tag{5}$$

where $\Lambda \equiv r^2\nabla\cdot\left(\frac{\nabla}{r^2}\right)r$. Adapting Pudritz (1981) for radial dependence of α and η due to shear induced turbulence to the case of galaxies (Mangalam & Subramanian, 1994, Sellwood & Balbus, 1999), we write $\eta = \dfrac{M^2h^2v_0}{r}$ and $\alpha = \dfrac{M^2hv_0}{r}$. Here M is the Mach number, h is the half-disc height and $v_0 = r\omega$ velocity of the disc.

DYNAMO EQUATIONS

We now introduce the following dimensionless numbers:
$R_\alpha = \frac{\alpha h}{\eta}$, $R_\omega = \frac{h^2 \omega}{\eta}$ and $D = R_\alpha R_\omega = \frac{\alpha h^3 \omega}{r^2}$.

And rescale the variables as
$r = \tilde{r}h$, $z = \tilde{z}h$, $T = \tilde{T}/h$, $t = \tilde{t}/\tau$ and $\Lambda = \tilde{\Lambda}/h^2$. Dropping the tilde, we can rewrite the dynamo equation in a more compact and symmetric manner as

$$\left(\frac{\partial}{\partial t} - \eta\Lambda\right)\psi = R_\alpha T \tag{6}$$

$$\left(\frac{\partial}{\partial t} - \eta\Lambda\right)T = R_\omega \frac{\partial \psi}{\partial z}. \tag{7}$$

Separation of Variables

We look for separable solution of the form

$$\psi(r,z) = \sum_{n=1}^{N} Q_n(r) a_n(z) \exp \Gamma t, \quad T(r,z) = \sum_{n=1}^{N} Q_n(r) b_n(z) \exp \Gamma t \tag{8}$$

Separation of variables leads to the following set of differential equations

$$r\Gamma Q(r) + \frac{1}{r}\frac{\mathrm{d}Q(r)}{\mathrm{d}r} - \frac{\mathrm{d}^2 Q(r)}{\mathrm{d}r^2} = \gamma Q(r) \tag{9}$$

$$\frac{\mathrm{d}^2 a(z)}{\mathrm{d}z^2} + R_\alpha \alpha(z) b(z) = \gamma a(z)$$

$$\frac{\mathrm{d}^2 b(z)}{\mathrm{d}z^2} + R_\omega \frac{\mathrm{d}a(z)}{\mathrm{d}z} = \gamma b(z) \tag{10}$$

The radial solutions are given by

$$Q_n(r) = rJ_1(\sqrt{\gamma_n}r) \tag{11}$$

with the boundary condition that $\sqrt{\gamma_n}R$ is the nth Bessel zero at the boundary $r = R$. The equations for $a(z)$ and $b(z)$ can be coupled into the following fourth order equation

$$\frac{d^4a}{dz^4} - 2\gamma_n \frac{d^2a}{dz^2} - D\frac{da}{dz} + \gamma_n^2 a = 0 \tag{12}$$

which has to be solved numerically as an eigenvalue problem.

CORONAL FIELDS

The dynamo generated fields are matched to a force-free corona at the boundary of the galaxy. The generic form of the coronal linear force-free field is given by (Mangalam & Subramanian 1994)

$$\nabla \times \mathbf{B_c} = \mu \mathbf{B_c}; \quad \nabla \cdot \mathbf{B_c} = 0 \tag{13}$$

where μ is the force-free parameter. The general solution to the above equations are given by

$$\psi_c(r,z) = \sum_{n=1}^{N} e_n J_1(\sqrt{\gamma_n} r) \exp(-\sqrt{\gamma_n - \mu^2}|z|) \tag{14}$$

$$T_c(r,z) = \sum_{n=1}^{N} \mu e_n J_1(\sqrt{\gamma_n} r) \exp(-\sqrt{\gamma_n - \mu^2}|z|) \tag{15}$$

where the coefficients e_n are determined from the boundary conditions.

BOUNDARY CONDITIONS

We use the following boundary conditions at the disk surface ($z = 1$)

$$\psi[1] = 0, \quad \left[\frac{\partial \psi}{\partial z}\right](1) = 0, \quad b_n(1) = \mu a_n(1) \tag{16}$$

where the brackets indicate continuity of the fields. The equatorial boundary conditions depend on the symmetry of the solutions and are given as

- quadrupolar conditions: $\psi(0) = 0$, $\left[\frac{\partial T}{\partial z}\right](0) = 0$
- dipolar conditions: $\left[\frac{\partial \psi}{\partial z}\right](0) = 0$, $T(0) = 0$.

RESULTS

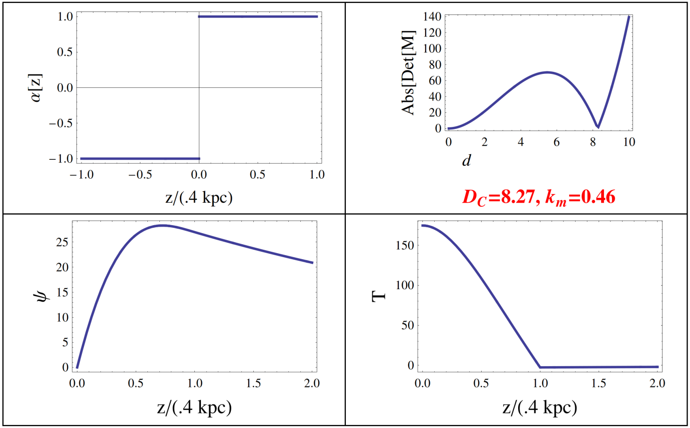


Figure : Top: Variation of α with height and a plot depicting the critical dynamo number. Bottom: Variation of poloidal current function T and Poloidal stream function ψ with height.

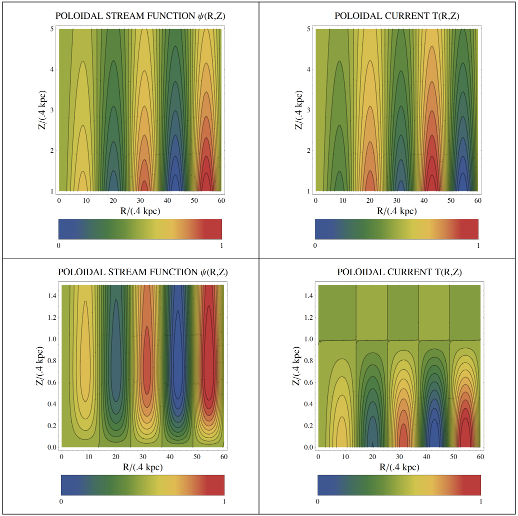


Figure : Meridional cross-sections of poloidal stream function ψ and poloidal current T as functions of r and z.

Expansion in Steady-State Solutions

In order to solve the time-dependent equations, we expand ψ and T as a linear combination of the Steady-state solutions with time dependent coefficients as

$$\psi(r,z,t) = \sum_{n=1}^{N} d_n(t) Q_n^s(r) a_n(z) \tag{17}$$

$$T(r,z,t) = \sum_{n=1}^{N} d_n(t) Q_n^s(r) b_n(z) \tag{18}$$

where $Q_n^s(r)$ is the steady-state radial solutions. $a_n(z)$ and $b_n(z)$ implicitly depend on time through α and μ.

Dynamo Equations

The radial part of the dynamo equation can now be expressed as

$$\sum_{n=1}^{N} r\frac{\dot{d}_n(t)}{d_n} - \frac{\lambda_r Q_n^s(r)}{Q_n^s(r)} = \sum_{n=1}^{N} \gamma_n, \tag{19}$$

where λ_r is the r part of the operator Λ. This can be simplified to

$$\langle J_m^s | r J_n^s \rangle \dot{d}_n - \langle J_m^s | J_n^s \rangle (\gamma_n - \gamma_n^s) d_n = 0 \tag{20}$$

where we have

$$\langle J_m^s | r J_n^s \rangle = \int_0^R r J_m^s J_n^s dr = \delta_{mn} \frac{R^2}{2} J_2^2(\sqrt{\gamma_n} R). \tag{21}$$

DYNAMICAL QUENCHING

The effect of the small-scale magnetic fields on the α effect can be described by writing (Shukurov et al. 2006, Sur et. al. 2007)

$$\alpha = \alpha_k + \alpha_m \tag{22}$$

$$\alpha_k = \frac{-1}{3}\overline{\tau \mathbf{U} \cdot \nabla \times \mathbf{U}}; \quad \alpha_m = \frac{1}{3\rho}\overline{\tau \mathbf{j} \cdot \mathbf{b}} \tag{23}$$

Here ρ is the fluid density and τ the correlation time of the turbulent velocity field. We further use the scaling relations and express α in terms of the small-scale helicity density:

$$\overline{\mathbf{j} \cdot \mathbf{b}} = k_f^2 \overline{\mathbf{a} \cdot \mathbf{b}}; \quad \alpha_m = \frac{1}{3}\tau \frac{k_f^2 \overline{\mathbf{a} \cdot \mathbf{b}}}{\rho}. \tag{24}$$

TIME DEPENDENCE OF α_m

Introducing a reference magnetic field strength and defining a magnetic Reynolds number as

$$B_{eq}^2 = \rho \bar{\mathbf{u}}^{\mathbf{2}}, \quad R_m = \frac{\eta_T}{\eta} \tag{25}$$

we can rewrite the small-scale helicity transport equation in terms of the time derivative of α as

$$\frac{d\alpha_m}{dt} = -2\eta_T k_f^2 \left(\frac{\alpha \langle \bar{B^2} \rangle - \eta_T \langle \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} \rangle}{B_{eq}^2} + \frac{\alpha_m}{R_m} \right) \tag{26}$$

where α_k is a constant and $R_m = \dfrac{\eta_T}{\eta}$.

MAGNETIC HELICITY IN CYLINDRICAL GEOMETRY

Magnetic Helicity is a measure of links between field lines.

$$H = -\frac{1}{4\pi}\int\int \mathbf{B}(\mathbf{x}) \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{B}(\mathbf{x}')d^3x d^3x' = \int \mathbf{A} \cdot \mathbf{B} d^3x \qquad (27)$$

The absolute helicity (Low 2006, 2011), is a gauge invariant flux-weighted sum of the writhe of that axial flux and its mutual linkage with the circulating flux which is conserved under condition of perfect electrical conductivity.

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_\psi + \mathbf{B}_\phi \\ \mathbf{B}_\psi &= \nabla \times \phi\hat{z}; \quad \mathbf{B}_\psi = \nabla \times \nabla \times \psi\hat{z} \end{aligned}$$

then the absolute helicity is

$$H_{abs}(\psi,\phi) = (\nabla \times \psi\hat{z}) \cdot [\nabla \times (\nabla \times \psi\hat{z}) + 2(\nabla \times \phi\hat{z})]. \qquad (28)$$

In our axisymmetric case, the above expression reduces to

$$H_{abs} = \langle \frac{2\psi T}{r^2} \rangle \qquad (29)$$

HELICITY CONSTRAINT EQUATION

The total magnetic helicity of the galactic disk + corona is conserved at any given time. This can be expressed as

$$H_d(t) + h_d(t) + H_c(t) = H_0. \tag{30}$$

The small scale helicity can be written in terms of α as $\langle h_d \rangle = \frac{B_{eq}^2}{k_f^2}\frac{\alpha_m}{\eta_T}$. Also, since for force-free fields $\psi_c = \mu T_c$, the helicity constraint equation can now be written as

$$\langle \frac{2\psi_d T_d}{r^2} \rangle + \langle \frac{B_{eq}^2}{k_f^2}\frac{\alpha_m}{\eta_T} \rangle + \langle \frac{2\mu\psi_c}{r^2} \rangle = H_0. \tag{31}$$

MAGNETIC HELICITY DISSIPATION

In a medium with finite resistivity, the rate of change of magnetic helicity can be calculated for small scale and large scale helicity as (Mangalam 2008)

$$\begin{aligned}
\left\langle \frac{dH_d}{dt} \right\rangle &= -2 \left\langle \eta \overline{\mathbf{J} \cdot \mathbf{B}} \right\rangle + \left\langle 2\boldsymbol{\varepsilon_T} \cdot \bar{\mathbf{B}} \right\rangle + \left[\frac{dH_d}{dt} \right]_s & (32) \\
\left\langle \frac{dh_d}{dt} \right\rangle &= -2 \left\langle \eta \overline{\mathbf{j} \cdot \mathbf{b}} \right\rangle - \left\langle 2\boldsymbol{\varepsilon_T} \cdot \bar{\mathbf{B}} \right\rangle + \left[\frac{dh_d}{dt} \right]_s & (33)
\end{aligned}$$

The helicity transprt terms are included through the change in helicity of the corona:

$$(dh_d/dt)_s + (dH_d/dt)_s = -dH_c/dt \qquad (34)$$

where the overall balance is expressed by the parameter μ of the corona used in the helicity.

COMPLETE TIME DEPENDENT FORMULATION

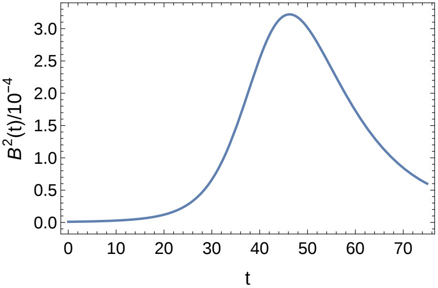
The complete set of equations to be solved simultaneously for the time-dependent problem can thus be listed as

1. $$\langle J_m^s | r J_n^s \rangle \dot{d}_n = \langle J_m^s | J_n^s \rangle (\gamma_n - \gamma_n^s) d_n$$

2. $$\frac{d\alpha_m}{dt} = -2\eta_T k_f^2 \left(\frac{\alpha \langle \bar{B}^2 \rangle - \eta_T \langle \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} \rangle}{B_{eq}^2} + \frac{\alpha_m}{R_m} \right)$$

3. $$\langle \frac{dH_c}{dt} \rangle = 2 \frac{\eta_T}{R_m} \left(\langle \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} \rangle + \frac{B_{eq}^2}{\eta_T} \alpha_m \right)$$

Preliminary Results



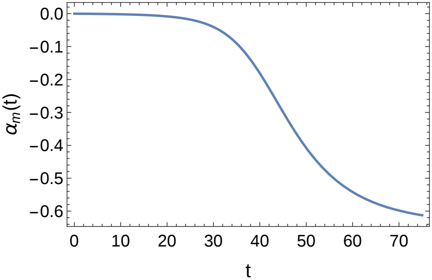


Figure : Variation of total energy and α_m with time.

Preliminary Results

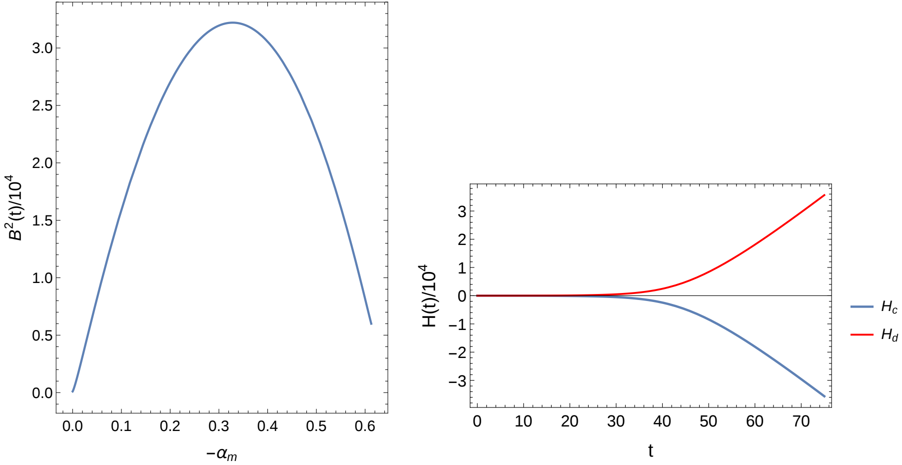


Figure : a. Parametric plot of total energy and α_m as a function of time. b. evolution of helicity in the disk and corona with time

Summary and Conclusions

- We have obtained steady-state solutions to the kinematic dynamo equations for the galactic magnetic field which are matched to a force-free corona outside the galactic disc. A critical dynamo number of $D_c \approx 8.26$, was found for the steady state solution.
- It is shown that the absolute helicity may calculated in cylindrical geometry using $h_{abs} = \int \frac{\psi T}{r^2}\, \mathrm{d}^3x$ using the prescription given in Low (2011).
- We have also developed a time-dependent formalism for the evolution of the magnetic field in which the disk dynamo is allowed to operate by transferring the magnetic helicity from the disk to the corona through a relaxation or a flux transport process.
- A preliminary study of the time-dependent solutions shows that the helicity conservation leads to suppression of α effect and leads to quenching of the dynamo.

FUTURE WORK

- To add flux terms in the helicity transport equation and study its effect on dynamo growth.
- To study the dependence of the saturated value and structure of the final field on the parameters α and μ.
- To incorporate the turbulence spectrum in the initial conditions for the expansion coefficients d_n.

References

1. Mangalam, A. V., & Subramanian, K. 1994, Apj, 434, 509
2. Mangalam, A. 2008, Turbulence, Dynamos, Accretion Disks, Pulsars and Collective Plasma Processes, 69
3. Low, B. C. 2006, Apj, 646, 1288
4. Low, B. C. 2011, Physics of Plasmas, 18, 052901
5. Pudritz, R. E. 1981, MNRAS, 195, 881
6. Sellwood, J. A. & Balbus, S. A.: 1999, ApJ 511 , 660
7. Shukurov A., Sokoloff D., Subramanian K., Brandenburg A., 2006, AA,448, L33
8. Sur, S., Shukurov, A., & Subramanian, K. 2007, MNRAS, 377, 874

FLUX PROPORTIONAL TO CORONAL HELICITY

We make the assumption that the helicity transport term is proportional to the helicity in the corona, i.e. $\left[\frac{dh}{dt}\right]_S = k\frac{\eta_T k_f^2}{B_{eq}^2}Hc$, where k is a constant and transfers has equal and opposite sign for large and small scales similar to $\boldsymbol{\varepsilon_T} \cdot \bar{\mathbf{B}}$. Then we have the following equations

$$
\begin{aligned}
\langle J_m^s | r J_n^s \rangle \dot{d}_n &= \langle J_m^s | J_n^s \rangle (\gamma_n - \gamma_n^s) d_n \\
\frac{d\alpha_m}{dt} &= -2\eta_T k_f^2 \left(\frac{\alpha \langle \bar{B^2} \rangle - \eta_T \langle \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} \rangle}{B_{eq}^2} + \frac{\alpha_m}{R_m} \right) + k\frac{\eta_T k_f^2}{B_{eq}^2} Hc \\
\langle \frac{dH_c}{dt} \rangle &= 2\frac{\eta_T}{R_m} \left(\langle \bar{\mathbf{J}} \cdot \bar{\mathbf{B}} \rangle + \frac{B_{eq}^2}{\eta_T} \alpha_m \right)
\end{aligned}
$$

PRELIMINARY RESULTS (WITH HELICITY FLUX)

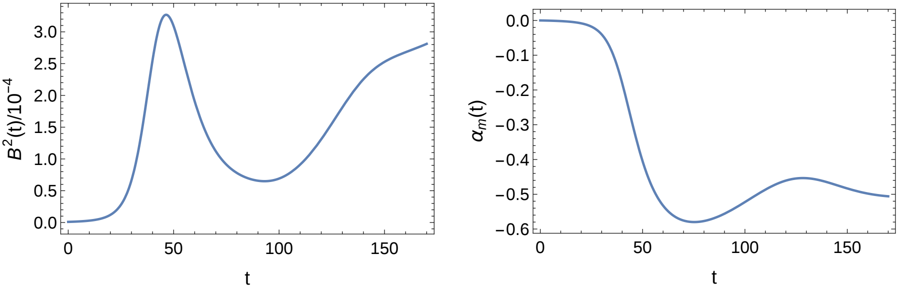
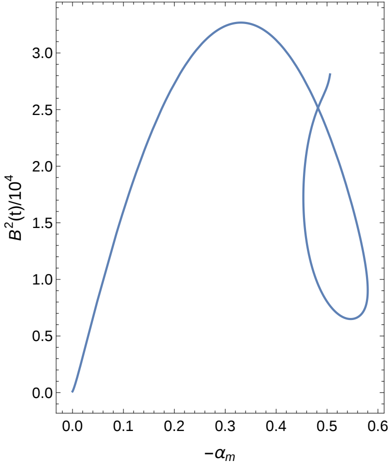


Figure : Variation of total energy and α_m with time.

PRELIMINARY RESULTS (WITH HELICITY FLUX)



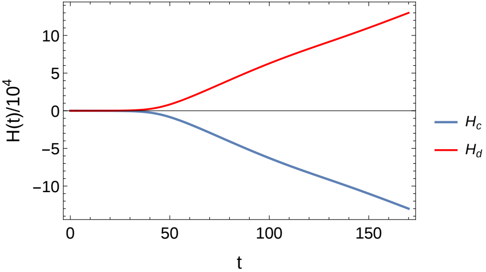


Figure : a. Parametric plot of total energy and α_m as a function of time. b. Evolution of the total helicity in the disk and corona with time